Explicit Spectral Analysis for Operators Representing the unitary group $\mathbb{U}(d)$ and its Lie algebra $\mathfrak{u}(d)$ through the Metaplectic Representation and Weyl Quantization

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Abstract

Let $A = \left(\begin{smallmatrix} B & C \\ -C & B \end{smallmatrix}\right)$, where B and C are $d \times d$ real matrices such that $B^* = -B$ and $C^* = C$. The main purpose of this talk is to compute and analyze the spectrum of the following family of operators on $L^2(\mathbb{R}^d)$ with domain $S(\mathbb{R}^d)$ (i.e. the Schwartz space):

$$H_A = \frac{1}{2}\sum C_{jk}(-\frac{\partial^2}{\partial x_j \partial x_k} + x_j x_k) + \frac{i}{2}\sum B_{jk}(x_k \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_k}) - \frac{1}{2}\mathrm{tr}(B).$$

It turns out that, under the identification $\mathbb{R}^{2d} \ni (x, \xi) \mapsto x + i\xi \in \mathbb{C}^d$, the matrices A of the previously described type correspond with matrices belonging to the Lie algebra $\mathfrak{u}(d)$ of anti-hermitian matrices and the map $A \mapsto H_A$ is a Lie algebra homomorphism. Recall that any anti-hermitian matrix has purely imaginary eigenvalues. Our first goal in this talk is to explain how we showed the following result.

Theorem 1 *Let $is_1, is_2, \cdots is_d$ be the eigenvalues of A. Then the point spectrum of H_A is given by*

$$\sigma_p(H_A) = \{-\sum s_j n_j \mid n \in \mathbb{N}_0^d\} + \frac{i}{2}tr(A)$$

and the spectrum of H_A is $\sigma(H_A) = \overline{\sigma_p(H_A)}$.

One of the main tools to prove the previous result is to notice that $H_A = -i\frac{d}{dt}\mu(e^{tA}) \mid_{t=0}$, where μ is the so called metaplectic representation. Moreover, we will show a similar expression for the spectrum of $\mu(U)$, for any $U \in \mathbb{U}(d)$.

We will give some simple conditions on the eigenvalues of A to guarantee that the spectrum of H_A is discrete. Under those conditions, using some combinatorial tools, we will prove that the multiplicity function is in certain sense a quasi-polynomial of degree $d-1$. Furthermore, we will show that the counting of eigenvalue function behaves like a so called Ehrhart polynomial of degree d. The latter result will lead us to a Weyl's law for the operators H_A.

References

[1] F. Belmonte and S. Cuellar, *Constants of motion of the Harmonic Oscillator*. Math. Phys. Anal. Geom. **23**, no.4 (2020), paper No. 35, 22 pp.