Explicit Spectral Analysis for Operators Representing the unitary group $\mathbb{U}(d)$ and its Lie algebra $\mathfrak{u}(d)$ through the Metaplectic Representation and Weyl Quantization

Fabián Belmonte
joint work with Giuseppe de Nittis

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THE OPERATORS REPRESENTING $\mathfrak{u}(d)$

- Let $A = \begin{pmatrix} B & C \\ -C & B \end{pmatrix}$, where B and C are $d \times d$ real matrices such that $B^* = -B$ and $C^* = C$. The main purpose of this talk is to compute and analyze the spectrum of the following family of operators on $L^2(\mathbb{R}^d)$ with domain $S(\mathbb{R}^d)$ (i.e. the Schwartz space):

$$H_A = \frac{1}{2}\sum C_{jk}\left(-\frac{\partial^2}{\partial x_j \partial x_k} + x_j x_k\right) + \frac{i}{2}\sum B_{jk}\left(x_k \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_k}\right) - \frac{1}{2}\mathrm{tr}(B).$$

- It turns out that, under the identification $\mathbb{R}^{2d} \ni (x,\xi) \mapsto x + i\xi \in \mathbb{C}^d$, the matrices A of the previously described type correspond with matrices belonging to the Lie algebra $\mathfrak{u}(d)$ of anti-hermitian matrices and the map $A \mapsto H_A$ is a Lie algebra homomorphism.

COMPUTING THE SPECTRA

Recall that any anti-hermitian matrix has purely imaginary eigenvalues. Our first purpose is to explain the following result.

THEOREM

Let $is_1, is_2, \cdots is_d$ be the eigenvalues of A. Then the point spectrum of H_A is given by

$$\sigma_p(H_A) = \{-\sum s_j n_j \mid n \in \mathbb{N}_0^d\} + \frac{i}{2} tr(A)$$

and the spectrum of H_A is $\sigma(H_A) = \overline{\sigma_p(H_A)}$.

After showing this result, we will look for conditions to guarantee that the spectrum is discrete. Under those conditions, we will use some combinatorial tools to study the multiplicity function. We will show that the counting of eigenvalue function behaves as a so called Ehrhart polynomial of degree d. Finally, using the Weyl symbol of H_A, we will show a Weyl's law for those operators.

Weyl Quantization and the Metaplectic Representation

One of the main tools to show our results is the Weyl quantization, which we denote by $\mathfrak{Op}^{\hbar}$. More precisely, for each $f \in S'(\mathbb{R}^{2d})$, let $\mathfrak{Op}(f) : S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ given by

$$[\mathfrak{Op}^{\hbar}(f)u](x) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{\hbar}(x-y)\cdot\xi} u(y) \mathrm{d}\xi \mathrm{d}y.$$

We also define $\mathfrak{Op} = \mathfrak{Op}^1$. It is straightforward to show $H_A = \mathfrak{Op}(p_A)$, where

$$p_A(w) = -\frac{1}{2}\langle w, A\mathfrak{J}w\rangle,$$

and $\mathfrak{J}$ is the so called symplectic matrix, i.e. $\mathfrak{J} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

- Let $Sp(d)$ be the real symplectic group. The metaplectic representation is a map $\mu_\hbar : Sp(d) \to \mathcal{U}(L^2(\mathbb{R}^d))$ such that

$$\mathfrak{Op}^\hbar(f \circ S^*) = \mu_\hbar(S)\mathfrak{Op}^\hbar(f)\mu_\hbar(S)^{-1},$$

for every $S \in Sp(d)$ and $f \in S'(\mathbb{R}^{2d})$.

- Despite its name, $\mu_\hbar$ is not a representation of $Sp(d)$ in the usual sense. However, this issue vanishes when $\mu_\hbar$ is restricted to $\mathbb{U}(d)$.
- It turns out that $H_A = -i\frac{d}{dt}\mu(e^{tA})|_{t=0}$.

Preservation of Constants of Motion of the Harmonic Oscillator

- Let $h_0(x,\xi) = 1/2(\|\xi\|^2 + \|x\|^2)$ be the classical harmonic oscillator and $H_0 = \frac{1}{2}(-\Delta + \|x\|^2)$ be the quantum harmonic oscillator.
- We will say that $f \in C^\infty(\mathbb{R}^{2d})$ is a (classical) constants of motion of h_0 if $\{h_0, f\} = 0$. It is straightforward to extend this notion to $f \in S'(\mathbb{R}^{2d})$.

Theorem

Let f be a real tempered constant of motion of the classical harmonic oscillator. Then $\mathfrak{Op}(f)[S(\mathbb{R}^d)] \subseteq S(\mathbb{R}^d)$. Moreover, $\mathfrak{Op}(f)$ with domain $S(\mathbb{R}^d)$ is an essentially selfadjoint operator on $L^2(\mathbb{R}^d)$ and we also denote by $\mathfrak{Op}(f)$ its closure. Furthermore, $\mathfrak{Op}(f)$ strongly commutes with H_0.

- Let $\{\phi_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ be the Hermite basis. It is well known that the eigenspaces of H_0 are $\mathcal{H}_k = \mathrm{span}\{\phi_\alpha \mid |\alpha| = k\}$, with $k \in \mathbb{N}_0$.
- In particular, if $f \in S'(\mathbb{R}^{2d})$ is a tempered constant of motion and we denote by $\mathfrak{Op}_k(f) := \mathfrak{Op}(f)|_{\mathcal{H}_k}$, then

$$\sigma_p(\mathfrak{Op}(f)) = \bigcup_k \sigma_p(\mathfrak{Op}_k(f))$$

  and $\overline{\sigma_p(\mathfrak{Op}(f))} = \sigma(\mathfrak{Op}(f))$.
- It is easy to show that p_A is a classical constant of motion of h_0.
- If σ_t denotes the Hamiltonian flow of h_0 and $g \in \mathbb{U}(d)$, then $g\sigma_t = \sigma_t g$. Moreover, $\mu(\sigma_t) = e^{itH_0}$. Therefore, $\mu(g)$ also commutes with H_0 and its spectrum admits the same decomposition than above.

BARGMANN TRANSFORM AND THE SYMMETRIC TENSOR PRODUCT

- Let $\mathfrak{F}^d$ be the Segal-Bargmann space, i.e. $\mathfrak{F}^d$ is the Hilbert space formed by all the holomorphic functions $F : \mathbb{C}^d \to \mathbb{C}$ such that $\int |F(z)|^2 e^{-\pi |z|^2} dz < \infty$. Also, let $B : L^2(\mathbb{R}^d) \to \mathfrak{F}^d$ be the Bargmann transform.
- It is well known that $B(\mathcal{H}_k)$ is the space spanned by the monomial of total degree k.
- For each $g \in \mathbb{U}(d)$, define $\nu(g) = B\mu(g)B^*$. Then, for each $q \in \mathfrak{F}^d$, we have that

$$[\nu(g)q](z) = \det(g)^{-1/2} q(g^{-1}z).$$

It is well known that $B(\mathcal{H}_k)$ is isomorphic with k-the symmetric power $S^k(\mathbb{C}^d)$ of $\mathbb{C}^d$. More precisely, let $e_1, \cdots e_d$ be the canonical basis of $\mathbb{C}^d$ and for each $j_1, \cdots j_k \in \{1, \cdots, d\}$, let $q_{j_1,\cdots j_k}(z) = z_{j_1} \cdots z_{j_k}$. It is well know that the map $T_k : S^k(\mathbb{C}^d) \to \mathfrak{F}_k^d$ given by

$$T_k(e_{j_1} \odot \cdots \odot e_{j_k}) = q_{j_1,\cdots j_k}$$

extends to an isomorphism of vector spaces, where $\odot$ denotes the symmetric tensor product.

Proposition

Let $\nu_k = \nu|_{B(\mathcal{H}_k)}$ *and* $\eta_k(g) := T_k^{-1}\nu_k(g)T_k : S^k(\mathbb{C}^d) \to S^k(\mathbb{C}^d)$. *Then*

$$\eta_k(g)(v_1 \odot v_2 \odot \cdots \odot v_k) = \det(g)^{-1/2}(\overline{g}v_1) \odot (\overline{g}v_2) \odot \cdots \odot (\overline{g}v_k),$$

for any $g \in \mathbb{U}(d)$ *and* $v_1, v_2, \cdots v_k \in \mathbb{C}^d$.

Computing the spectrum of $\mu(g)$

Theorem

Let $\theta_1, \cdots, \theta_d$ the eigenvalues of $g \in \mathbb{U}(d)$. Then, the point spectrum of $\mu(g)$ is given by

$$\sigma_p(\mu(g)) = \det(g)^{-1/2} \cdot \{\overline{\theta}^n := \overline{\theta}_1^{n_1} \cdots \overline{\theta}_d^{n_d} \mid (n_1, \cdots, n_d) \in \mathbb{N}_0^d\}.$$

For any $g \in \mathbb{U}(d)$, g has an irrational rotation eigenvalue if and only if $\sigma(\mu(g)) = \mathbb{S}^1$. Moreover, if $\theta_j = exp\left(\frac{2\pi i p_j}{q_j}\right)$ with $p_j \in \mathbb{Z}$ and $q_j \in \mathbb{N}$, then

$$\sigma_p(\mu(g)) = \sigma(\mu(g)) = \det(g)^{-1/2} \cdot \left\{ exp(\frac{2\pi i n p}{q}) \mid n \in \mathbb{N} \right\},$$

where q is the least common multiple of the denominators $q_1, \cdots, q_d$ and p is the greatest common divisor of $\frac{q|p_1|}{q_1}, \cdots, \frac{q|p_d|}{q_d}$. In particular, $\sigma(\mu(g))$ is a translation of a finite subgroup of $\mathbb{S}^1$.

Sketch of the proof: Let $v_1, \cdots v_d$ a basis of eigenvalues corresponding to $\theta_1, \cdots, \theta_d$. Thus,

$$\begin{aligned}\eta_k(g)(\overline{v_{j_1}} \odot \cdots \odot \overline{v_{j_k}}) &= \det(g)^{-1/2}(\overline{g v_{j_1}}) \odot \cdots \odot (\overline{g v_{j_k}}) \\ &= \det(g)^{-1/2} \cdot \overline{\theta}_{j_1} \cdots \overline{\theta}_{j_k} (\overline{v_{j_1}} \odot \cdots \odot \overline{v_{j_k}})\end{aligned}$$

Since the vectors $(\overline{v_{j_1}} \odot \cdots \odot \overline{v_{j_k}})$ forms a basis of $S^k(\mathbb{C}^d)$, the first part of our result follows.

Sketch of the proof of our first result: Taking $g = e^{tA}$ in the previous proposition and derivating at $t = 0$ we obtain that

$$\begin{aligned}& d\eta_k(A)(v_1 \odot v_2 \odot \cdots \odot v_k) \\ &\quad = -\frac{1}{2}\mathrm{tr}(A)(v_1 \odot v_2 \odot \cdots \odot v_k) + (\overline{A} v_1) \odot v_2 \odot \cdots \odot v_k \\ &\quad + v_1 \odot (\overline{A} v_2) \odot \cdots \odot v_k + \cdots + v_1 \odot v_2 \odot \cdots \odot (\overline{A} v_k)\end{aligned}$$

Let $\{v_j\}$ be a basis of $\mathbb{C}^d$ such that $Av_j = is_j v_j$. Then

$$\begin{aligned} d\eta_k(A)(\overline{v_{j_1}} \odot \cdots \odot \overline{v_{j_k}}) &= -\frac{1}{2}\mathrm{tr}(A)(\overline{v_{j_1}} \odot \cdots \odot \overline{v_{j_k}}) \\ &+ (\overline{Av_{j_1}}) \odot v_2 \odot \cdots \odot \overline{v_{j_k}} + \cdots \\ &+ \overline{v_{j_1}} \odot \cdots \odot (\overline{Av_{j_k}}) \\ &= \left(-\frac{1}{2}\mathrm{tr}(A) - i \sum_{l=1}^{k} s_{j_l} \right) (\overline{v_{j_1}} \odot \cdots \odot \overline{v_{j_k}}). \end{aligned}$$

Since the vectors $(\overline{v_{j_1}} \odot \cdots \odot \overline{v_{j_k}})$ forms a basis of $S^k(\mathbb{C}^d)$, our result follows.

DISCRETE SPECTRUM AND MULTIPLICITY

PROPOSITION

Let $is_1, is_2, \cdots is_d$ be the eigenvalues of A and $L(s) = \{-\sum s_j n_j \mid n \in \mathbb{N}_0^d\}$.. The following statements are equivalent:

- A) *The subgroup of $\mathbb{R}$ generated by the monoid $L(s)$ is of the form $x\mathbb{Z}$, for some $x \in \mathbb{R}$.*
- B) *There is $x \in \mathbb{R}$ and $p_1, \cdots p_d \in \mathbb{Z}$ such that $s_j = r_j x$.*
- C) *$\sigma_p(H_A)$ is uniformly topologically discrete, i.e. there is $r > 0$ such that $(\lambda - r, \lambda + r) \cap (\zeta - r, \zeta + r) = \emptyset$, for every $\lambda, \zeta \in \sigma_p(H_A)$.*

If any of the previous statements holds then $\sigma(H_A) = \sigma_p(H_A)$.

For each $\lambda \in \sigma_p(H_A)$, let

$$M_\lambda = \left\{ (n_1, n_2, \cdots, n_d) \in \mathbb{N}_0^d \;\middle|\; -\sum n_j s_j + \frac{i}{2}\mathrm{tr}(A) = \lambda \right\}$$

From the proof of our first theorem it follows that the multiplicity of λ is $m_A(\lambda) = \#(M_\lambda)$. This, together with some known combinatorial results, implies the following result.

Proposition

Let $A \in \mathfrak{u}(d)$ and $is_1, is_2, \cdots, is_d$ its eigenvalues. Assume that $s_j = p_j x$ with $p_j \in \mathbb{Z}$ and $x \in \mathbb{R} - \{0\}$, for each $1 \leq j \leq d$. All the eigenvalues of H_A have finite multiplicity if and only if the real numbers $s_1, \cdots, s_d$ have the same sign. In such case, if $m_A(\lambda)$ denotes the multiplicity of the eigenvalue λ, then

$$m_A(\lambda) = \sum_{j=1}^{d} a_j(|(\lambda - \frac{i}{2} tr(A))x^{-1}|)\lambda^{j-1},$$

where $a_j(k)$ depends only of residues of k moduli $d!$, for each $k \in \mathbb{N}$ and $1 \leq j \leq d$.

For simplicity, in what follows we will assume that $s_j = p_j x < 0$, and we will choose $x > 0$. Let

$$N_A(r) = \#\{\lambda \in \sigma(H_A) \mid \lambda \leq r\} = \#\{n \in \mathbb{N}_0^d \mid -\sum n_j s_j + \frac{i}{2}\mathrm{tr}(A) \leq r\}$$

and

$$\mathcal{P} = \{x \in \mathbb{R}^d \mid x \geq 0, -\sum x_j p_j \leq q\},$$

where q be the minimal common multiple of $-p_1, -p_2, \cdots - p_d$. It is straightforward to show that, if $k = q^{-1}x^{-1}(r - \frac{i}{2}\mathrm{tr}(A)) \in \mathbb{N}_0$, then

$$i(\mathcal{P}, k) := \#\left(k\mathcal{P} \bigcap \mathbb{Z}^d\right) = N_A(r).$$

E. Ehrhart proved that the left hand side is a polynomial for any Polyhedra $\mathcal{P}$ with integer vertices and this was the beginning of an important topic in combinatorics. Some of the results on that topic lead to the following result.

THEOREM (SPECTRAL ASYMPTOTICS)

Let $is_1, is_2, \cdots is_d$ be the eigenvalues of $A \in \mathfrak{u}(d)$. Assume that $s_j < 0$ and that there are $p_j \in \mathbb{Z}$ and $x \in \mathbb{R}$ such that $s_j = p_j x$, for each $1 \leq j \leq d$. Choose $x > 0$ and let q be the minimal common multiple of $-p_1, -p_2, \cdots - p_d$. Then there is a polynomial $p(k) = \sum_{j=0}^{d} c_j k^j$ such that

$$p\left([(r - \frac{i}{2}tr(A))(qx)^{-1}]\right) \leq N_A(r) \leq p\left([(r - \frac{i}{2}tr(A))(qx)^{-1}] + 1\right),$$

where $[t]$ denotes the integer part of t, for any $t \in \mathbb{R}$. The inequality in the left hand side becomes an equality whenever $\frac{r - \frac{i}{2}tr(A)}{qx} \in \mathbb{N}_0$. Moreover, if $\mathcal{P} = \{x \in \mathbb{R}^d \mid x \geq 0, -\sum x_j p_j \leq q\}$, then $c_0 = 1$, $c_d = |\mathcal{P}|$ is the volume of $\mathcal{P}$ and $c_{d-1} = \frac{1}{2}|\partial \mathcal{P}|$ is one half of the sum of the $(d-1)$-volume of the faces of $\mathcal{P}$.

Introducing Planck's constant dependence, it is not difficult to show that $H_A^\hbar := \mathfrak{O}\mathfrak{p}^\hbar(p_A)$ is unitary equivalent to $\hbar H_A$ and the spectral analysis above can be repeated. The latter result implies the following.

Theorem (Weyl's law)

Let $is_1, is_2, \cdots is_d$ be the eigenvalues of $A \in \mathfrak{u}(d)$. Assume that $s_j < 0$ and that there are $p_j \in \mathbb{Z}$ and $x \in \mathbb{R}$ such that $s_j = p_j x$, for each $1 \leq j \leq d$. Also let $H_A^\hbar = \mathfrak{O}\mathfrak{p}^\hbar(p_A)$, $N_A^\hbar(r) = \#\{\lambda \in \sigma(H_A^\hbar) \mid \lambda \leq r\}$ and $\mathcal{E}_A(r) = \{(x,\xi) \in \mathbb{R}^{2d} \mid p_A(x,\xi) \leq r\}$. Then

$$N_A^\hbar(r) = (2\pi\hbar)^{-d}|\mathcal{E}_A(r)| + \frac{\|s\|}{2}(2\pi)^{-d}\hbar^{-d+1}\int_{\partial\mathcal{E}_A(r)} \|\nabla p_A\|^{-1} d\mu_r^A + O(\hbar^{-d+2}r^{d-2}).$$

where μ_r^A is the measure corresponding to the canonical volume form on $\partial\mathcal{E}_A(r) = \{(x,\xi) \in \mathbb{R}^{2d} \mid p_A(x,\xi) = r\}$.

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MUCHAS GRACIAS!!