Asintótica de valores propios para operadores de Dirac magnéticos en dos dimensiones

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Joint work with Vincent Bruneau

The magnetic Dirac operator in dimension two

We consider in $L^2(\mathbb{R}^2)^2$ the Dirac operator with a homogeneous magnetic field $B = (0,0,b)$. This operator is defined by

$$D_0 := \sigma \cdot (-i\nabla - A) + m\sigma_3$$

where $\sigma := (\sigma_1, \sigma_2)$, σ_3 are the Pauli matrices, $A := (A_1, A_2) = b(-\frac{x_2}{2}, \frac{x_1}{2})$.

More explicitly, by defining

$$a := (-i\partial_{x_1} - A_1) + i(-i\partial_{x_2} - A_2) = -2ie^{-b|x|^2/4}\overline{\partial}\, e^{b|x|^2/4}$$

$$a^* := (-i\partial_{x_1} - A_1) - i(-i\partial_{x_2} - A_2) = -2ie^{b|x|^2/4}\partial\, e^{-b|x|^2/4}$$

then

$$D_0 = \begin{pmatrix} m & a^* \\ a & -m \end{pmatrix}.$$

The operator D_0 is essentially self adjoint in $C_0^\infty(\mathbb{R}^2)^2$.

Relation with the Landau Hamiltonian

Define now the Landau Hamiltonian

$$H_L := (-i\nabla - A)^2.$$

It is easy to see that

$$D_0^2 = \begin{pmatrix} H_L - b + m^2 & 0 \\ 0 & H_L + b + m^2 \end{pmatrix}$$

The spectrum of H_L is given by the *Landau levels* $\Lambda_n = b(2n+1)$, $n \in \mathbb{Z}_+$. Moreover, by using the Foldy-Wouthuysen unitary transformation U_{FW}

$$D_0 = U_{FW}^* \begin{pmatrix} \sqrt{H_L - b + m^2} & 0 \\ 0 & -\sqrt{H_L + b + m^2} \end{pmatrix} U_{FW}.$$

Then, the spectrum of D_0 is made up of eigenvalues of infinite multiplicities, the so-called *Landau-Dirac* Levels

$$\mu_q := \begin{cases} \sqrt{2bq + m^2}, & q \in \{0, 1, 2, ...\} \\ -\sqrt{2b|q| + m^2}, & q \in \{-1, -2, ...\}. \end{cases}$$

Orthogonal projections

We have the relations $a^*a = H_L - b$ and $aa^* = a^*a + 2b = H_L + b$, for any $n \in \mathbb{Z}_+$, then

$$\mathrm{Ker}(H_L - \Lambda_n) = (a^*)^n \mathrm{Ker}(a).$$

Denote by $\boldsymbol{p_n}$ be the orthogonal projection onto $\mathrm{Ker}(H_L - \Lambda_n)$ $(n \in \mathbb{Z}_+)$

Similarly, denote by $\boldsymbol{\mathcal{P}_q}$ the orthogonal projection onto $\mathrm{Ker}(D_0 - \mu_q)$ $(q \in \mathbb{Z})$.

It is not dificult to see that

$$\mathcal{P}_q = \begin{cases} U_{FW}^* \begin{pmatrix} p_q & 0 \\ 0 & 0 \end{pmatrix} U_{FW}, & q \in \{0, 1, 2, ...\} \\ U_{FW}^* \begin{pmatrix} 0 & 0 \\ 0 & p_{|q|} \end{pmatrix} U_{FW}, & q \in \{-1, -2, ...\}. \end{cases}$$

Perturbation and eigenvalue counting function

Let V_1, V_2 and W be measurable decaying functions in $\mathbb{R}^2$ and take

$$V := \begin{pmatrix} V_1 & \overline{W} \\ W & V_2 \end{pmatrix}.$$

(If $W = -\widetilde{A}_1 - i\widetilde{A}_2$, and $\widetilde{b} = \partial_{x_1}\widetilde{A}_2 - \partial_{x_2}\widetilde{A}_1$, then the total magnetic field is $b + \tilde{b}$).

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This is a relatively compact perturbation of D_0, then the perturbed operator

$$D_V := D_0 + V$$

satisfies:

$$\sigma_{\mathrm{ess}}(D_V) = \sigma_{\mathrm{ess}}(D_0) = \{\mu_q,\ q \in \mathbb{Z}\}.$$

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Denote by $E_T(\omega)$ the spectral projection of the self-adjoint operator T associated with the Borel set ω.

For $q \in \mathbb{Z}$ set

$$\mathcal{N}_q^+(\lambda) := \mathrm{Tr} E_{D_0+V}(\mu_q + \lambda, \alpha), \qquad \mathcal{N}_q^-(\lambda) := \mathrm{Tr} E_{D_0+V}(\alpha, \mu_q - \lambda),$$

where α is a fixed number in (μ_q, μ_{q+1}) and (μ_{q-1}, μ_q), respectively.

Some history and methods for related results

On the asymptotic distribution of eigenvalues for the magnetic Schrödinger operator $H_L + v$:

- Raikov 1990, Ivrii 90's: v moderately decaying

$$v(x) \sim |x|^{-\gamma} \Longrightarrow \mathcal{N}_n(\lambda) \sim |\lambda|^{-2/\gamma}$$

  An important ingredient: The Toeplitz operator $p_n v p_n$.

- Raikov-Warzel 2002, Melgaard-Rozenblum 2003: v fast decaying

$$v(x) \text{ with compact support} \Longrightarrow \mathcal{N}_n(\lambda) \sim \frac{|\ln \lambda|}{\ln |\ln \lambda|}$$

- Filonov-Pushnitski 06: improvement

$$\mathcal{N}_n(\lambda) = \frac{|\ln \lambda|}{\ln |\ln \lambda|} + \frac{|\ln \lambda| \ln(\ln |\ln \lambda|)}{(\ln |\ln \lambda|)^2} + \frac{|\ln \lambda|}{(\ln |\ln \lambda|)^2} \big(\mathfrak{C} + o(1)\big)$$

  Both results need $v \geq C > 0$.

Problems with fast decaying perturbations
Symbols of variable sign

- Pushnitski, Rozenblum 2007, Persson 2009, Goffeng, Kachmar, Persson-Sundqvist 2016: obtacles
- Pushnitski, Rozenblum, 2011, results for some potentials with not fixed sign
- Lungenstrass, Raikov 2016, metric perturbations
- Cárdenas, Raikov, Tejeda 2020: non-local perturbations.
- Behrndt, Exner, Holzmann, Lotoreichik 2020: δ-potentials on curves

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On the asymptotic distribution of eigenvalues for magnetic Dirac operators

- Ivrii 90's: V power-like decaying
- Melgaard-Rozenblum 2003: $V = vI_2 \geq 0$ compactly suported

We are interested in V's of compact support with no fixed sign

First results for $\mathcal{N}_q^+$: Index of a pair of projections

For A and B self adjoint define

$$\begin{aligned}\Xi(\lambda; B, A) =& \dim \operatorname{Ker}\Big(E_A(-\infty,\lambda) - E_B(-\infty,\lambda) - I\Big)\\ &- \dim \operatorname{Ker}\Big(E_A(-\infty,\lambda) - E_B(-\infty,\lambda) + I\Big)\end{aligned}$$

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Lemma

Let $\lambda \notin \sigma_{\mathrm{ess}}(A)$. If $(B-A)$ is $A-$compact, then $E_B(-\infty,\lambda) - E_A(-\infty,\lambda)$ is compact

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$[\lambda_1,\lambda_2]\subset\mathbb{R}\setminus\sigma_{\mathrm{ess}}(A)\Rightarrow\Xi(\lambda_1;B,A)-\Xi(\lambda_2;B,A)=\mathcal{N}([\lambda_1,\lambda_2);B)-\mathcal{N}([\lambda_1,\lambda_2);A)$

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Lemma

If $M_1\geq M_2$ then $\Xi(\lambda;A,A+M_1)\geq\Xi(\lambda;A,A+M_2)$.

Diagonalization trick

For a given $\epsilon > 0$ define the potentials

$$V_\epsilon^\pm := \begin{pmatrix} V_1 \pm \epsilon(|V_1| + |W|) & W^* \\ W & V_2 \pm \epsilon(|V_2| + |W|) \end{pmatrix}$$

$$\mathcal{P}_q V_\epsilon^- \mathcal{P}_q + \mathcal{P}_q^\perp (V - \epsilon^{-1}|V|)\mathcal{P}_q^\perp \le V \le \mathcal{P}_q V_\epsilon^+ \mathcal{P}_q + \mathcal{P}_q^\perp (V + \epsilon^{-1}|V|) P_q^\perp$$

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Proposition

$$n_+(\lambda, \mathcal{P}_q V_\epsilon^- \mathcal{P}_q) + O(1) \leq \mathcal{N}_q^+(\lambda) \leq n_+(\lambda, \mathcal{P}_q V_\epsilon^+ \mathcal{P}_q) + O(1)$$

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For $q \ge 0$ set

$$\begin{aligned} T_0(V) &:= p_0 V_1 p_0; \\ T_q(V) &:= t_q p_q V_1 p_q + \frac{1 - t_q}{2bq} p_q a^* V_2 a p_q + \frac{1}{2\mu_q} p_q (a^* W + W^* a) p_q \end{aligned}$$

Lemma

$$\mathcal{P}_q V \mathcal{P}_q = U_{FW}^* \begin{pmatrix} T_q(V) & 0 \\ 0 & 0 \end{pmatrix} U_{FW}$$

Main result I: V_1 non negative

Theorem (1)

Let Ω be a bounded open set with Lipschitz boundary in $\mathbb{R}^2$. Assume $V_1, V_2, W \in L^\infty(\mathbb{R}^2)$ with support in $\overline{\Omega}$ and $V_1 \geq C > 0$. Then for any $q \in \mathbb{Z}$

$$\mathcal{N}_q^+(\lambda) = \frac{|\ln\lambda|}{\ln|\ln\lambda|} + \frac{|\ln\lambda|\ln(\ln|\ln\lambda|)}{(\ln|\ln\lambda|)^2} + \frac{|\ln\lambda|}{(\ln|\ln\lambda|)^2}\big(\mathfrak{C}(\Omega) + o(1)\big) \qquad \lambda \searrow 0.$$

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Here

$$\mathfrak{C}(\Omega) := 1 + \ln\left(\frac{b}{2}\,\mathrm{Cap}(\Omega)^2\right),$$

and $\mathrm{Cap}(\Omega)$ coincides with the *transfinite diameter*, i.e., is equal to $\lim_{n\to\infty}\delta_n(\Omega)$ where

$$\delta_n(\Omega) := \max_{z_1,\cdots,z_n\in\mathcal{E}}\left(\prod_{1\leq i<j\leq n}|z_i - z_j|\right)^{\frac{2}{n(n-1)}}.$$

Some ideas of the proof

The space $p_n L^2(\mathbb{R}^2)$ is isometric with the Fock space $\mathcal{F}^2$, i.e., the Hilbert space consisting of all entire functions f such that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-b|z|^2/4} dm(z) < \infty.$$

Introduce in $\mathcal{F}^2$ the quadratic forms

$$r_q(v)[f] := \int_{\mathbb{C}} |(a^*)^q e^{-b|z|^2/4} f(z)|^2 v(z) dm(z).$$

$$s_n(W)[f] = 2\mathrm{Re} \int_{\mathbb{C}} (a^*)^n f(z) \overline{(a^*)^{n-1} f(z)} e^{-b|z|^2/2} \overline{W}(z) dm(z).$$

Then $T_q(V)$ is unitarily equivalent to the operator in $\mathcal{F}^2$ given by the quadratic form

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Lemma

There exists a subspace of finite codimension in $\mathcal{F}^2$ where

$$r_q(V_1) + r_{q-1}(V_2) + s_q(W) \geq C r_0(\chi_\Omega)$$

Consider the case $q = 1$. From the Cauchy-Schwarz inequality we have the estimate, for any $\delta > 0$:

$$|s_1(W)[f]| \leq \delta a_1(|W|)[f] + \frac{1}{\delta} a_0(|W|)[f].$$

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Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, for any $\gamma>0$ there exists a subspace of finite codimension in $H^1(\Omega)$ such that $\|f\|_{L^2(\Omega)} \leq \gamma \|\nabla f\|_{L^2(\Omega)}$.

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Now we follow the same analysis of Filonov-Pushnitski'06 using orthogonal polynomials.

Main Result 2: The negative part is "encircled" by the positive part

We say that a compact set K is *encircled* by an open set Ω if there exists a Jordan curve $\Gamma \subset \Omega$ such that K is contained in the interior part of Γ.

Theorem

Let $V \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Suppose there exist $K \subset \mathbb{R}^2$ a compact set, and Ω_1, Ω_2 open bounded subsets of $\mathbb{R}^2$ such that K is encircled by $\Omega_1 \cup \Omega_2$. Further, assume

$$V_1 \geq C_1 \chi_{\Omega_1} - C\chi_K; \quad V_2 \geq C_2 \chi_{\Omega_2} - C\chi_K; \quad |W| \leq C_3 \chi_{\Omega_1} w + C\chi_K,$$

for some constants $C, C_1, C_2, C_3 \geq 0$.
Then the asymptotics of the previous theorem holds.

Gracias por su atención!