Crossings of classical Trajectories and Resonances for Schrödinger systems

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I/ Setting

Consider the 1D 2x2 matrix Schrödinger operator

$$P = P(h) = \begin{pmatrix} P_1(h) & hW \\ hW^* & P_2(h) \end{pmatrix} \quad \text{in } L^2(\mathbb{R};\mathbb{C}^2)$$

where

$$P_j(h) = (hD_x)^2 + V_j(x) \quad (j = 1,2)$$ with smooth $V_j \in C^\infty(\mathbb{R};\mathbb{R})$

and

$$W = r_0(x) + i\, r_1(x)\, hD_x \; : \text{Interaction}, \qquad r_0, r_1 \in C^\infty(\mathbb{R};\mathbb{R})$$

Here $D_x := -i\frac{d}{dx}$

$h > 0$ is a small parameter (the semiclassical parameter)

Classical dynamics

Let $p_j(x,\xi) = \xi^2 + V_j(x)$, $(x,\xi) \in T^*\mathbb{R} \simeq \mathbb{R}^2$, be the symbol of $P_j(h)$.

The classical dynamics is described by

$$H_{p_j} = \partial_\xi p_j \, \partial_x - \partial_x p_j \, \partial_\xi = 2\xi \, \partial_x - V_j'(x) \, \partial_\xi$$

and the classical trajectory is

$$(x(t), \xi(t)) = \exp(t H_{p_j})(x_0, \xi_0), \quad t \in \mathbb{R}.$$

Hamiltonian flow

In our 1D setting, it coincides with the characteristic set

$$\Gamma_j(E) := \{ (x,\xi) \in \mathbb{R}^2 ; \; p_j(x,\xi) = E \} \qquad (E \in \mathbb{R})$$

Conservation law: $p_j(\exp(tH_{p_j})(x,\xi)) = p(x,\xi)$

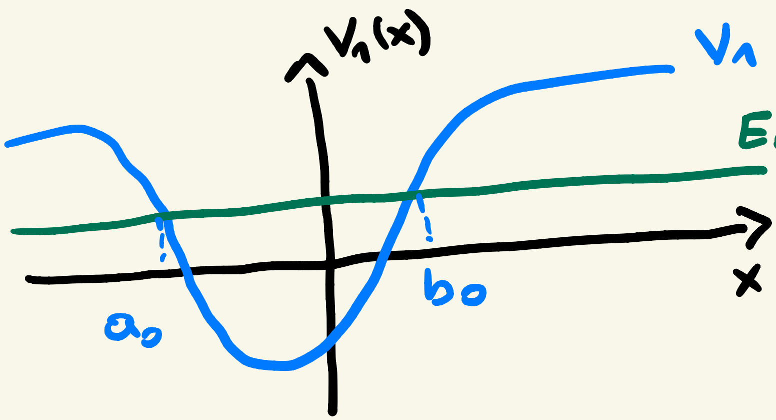
for all $(x,\xi) \in \Gamma_j(E)$.

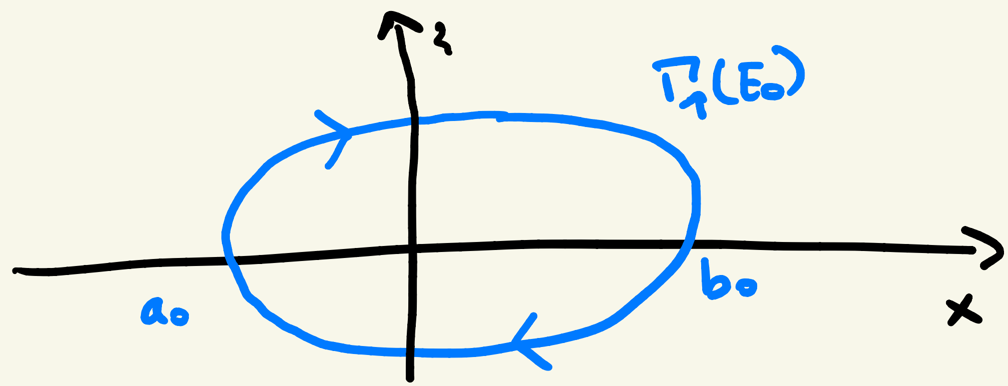
The model: Assume $V_j(x) \xrightarrow[x \to \pm\infty]{} V_{j,\pm} \in \mathbb{R}$.

Fix $E_0 \in \mathbb{R}$. We consider the following 2 types of potentials

(A1) V_1 is a simple well potential at E_0:

$V_{1,\pm} > E_0$ and $\exists\, a_0 < b_0$ s.t. $\dfrac{V_1(x) - E_0}{(x-a_0)(x-b_0)} > 0, \ \forall x \in \mathbb{R}$.





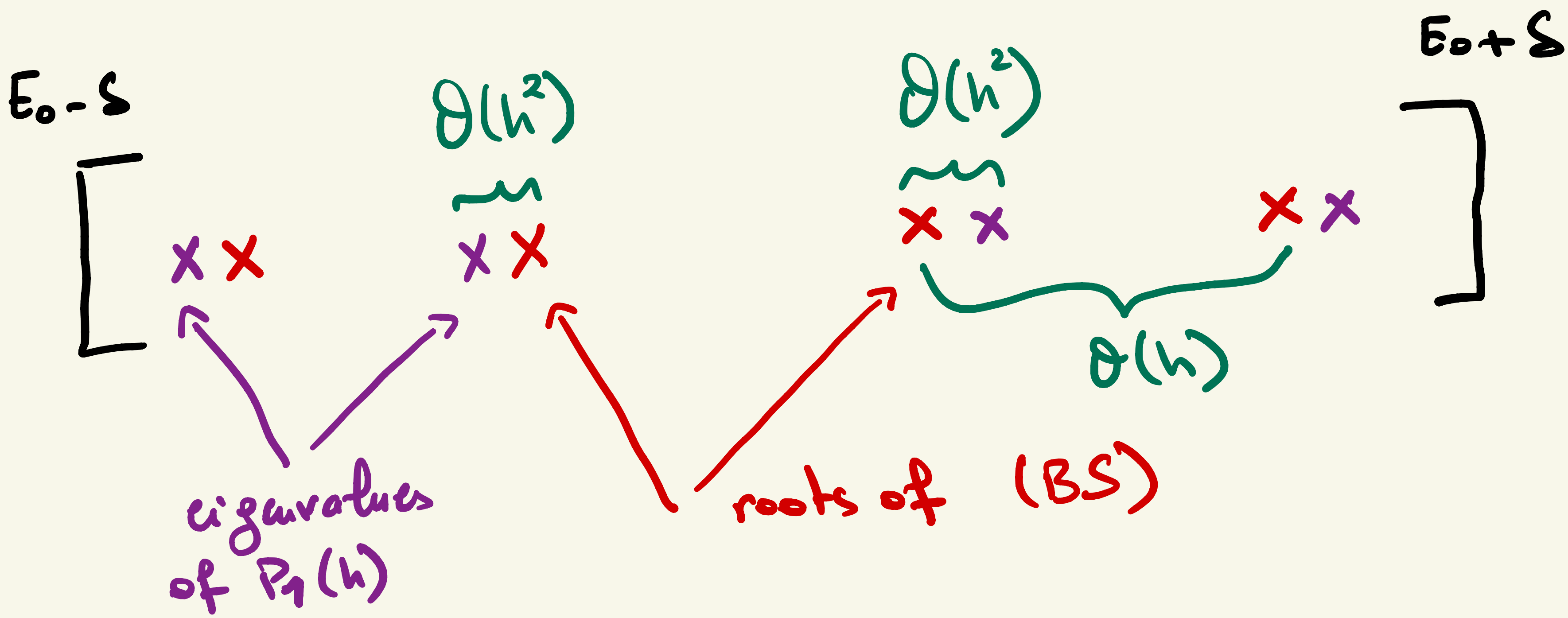
The spectrum of $P_1(h)$ near E_0 is discrete consists of simple h-dependent eigenvalues.

In the semiclassical limit $h \to 0^+$, these eigenvalues are approximated by the roots of

$$\cos\left(\frac{A(E)}{2h}\right) = 0$$

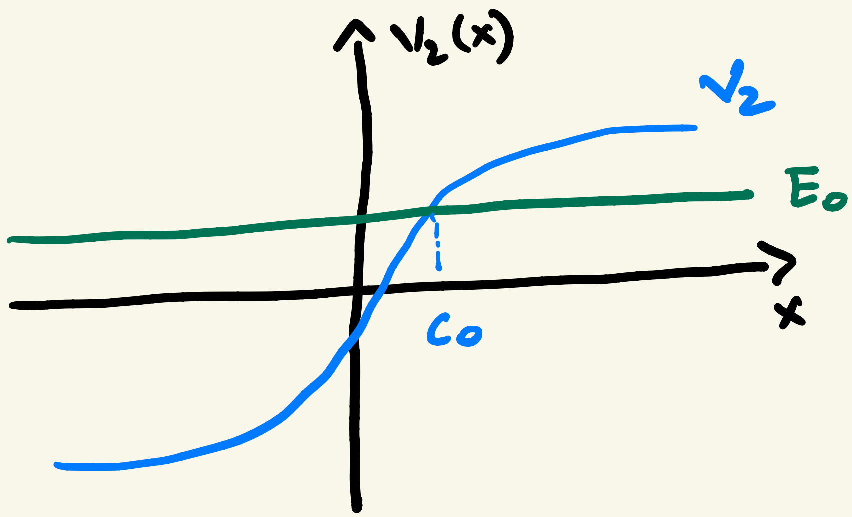
Bohr–Sommerfeld quantization rule (BS)

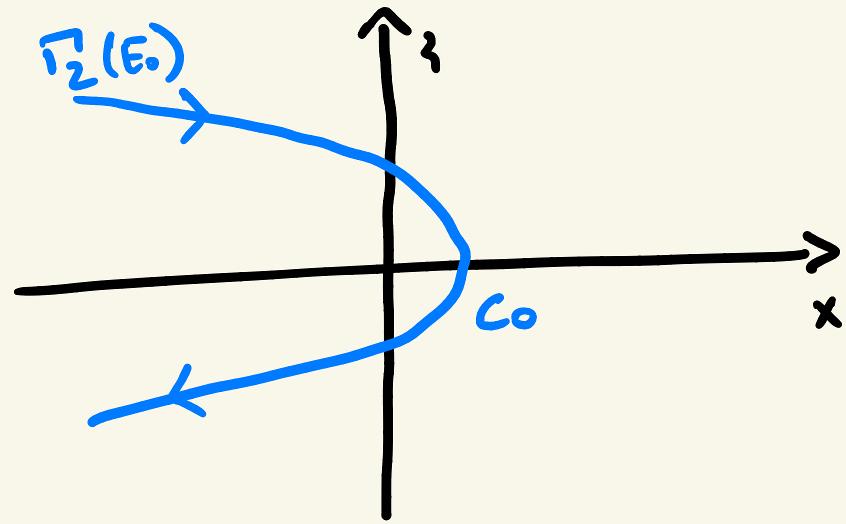
$$A(E) = \int_{\Gamma_1(E)} \xi \, dx$$: The action along $\Gamma_1(E)$



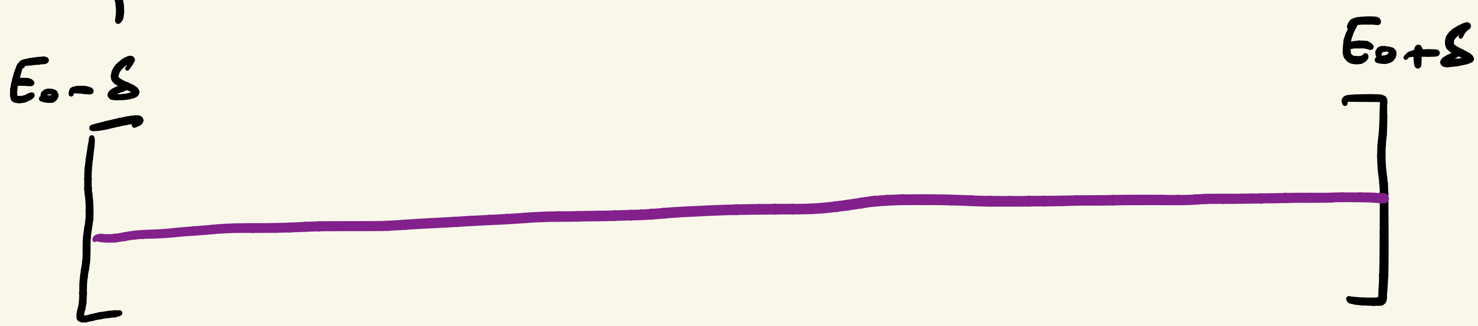
(A2) V_2 is a non-trapping potential at E_0:

$V_{2,-} < E_0 < V_{2,+}$ and $\exists\, c_0 \in \mathbb{R}$ s.t. $\dfrac{V_2(x) - E_0}{(x - c_0)} > 0, \ \forall x \in \mathbb{R}$



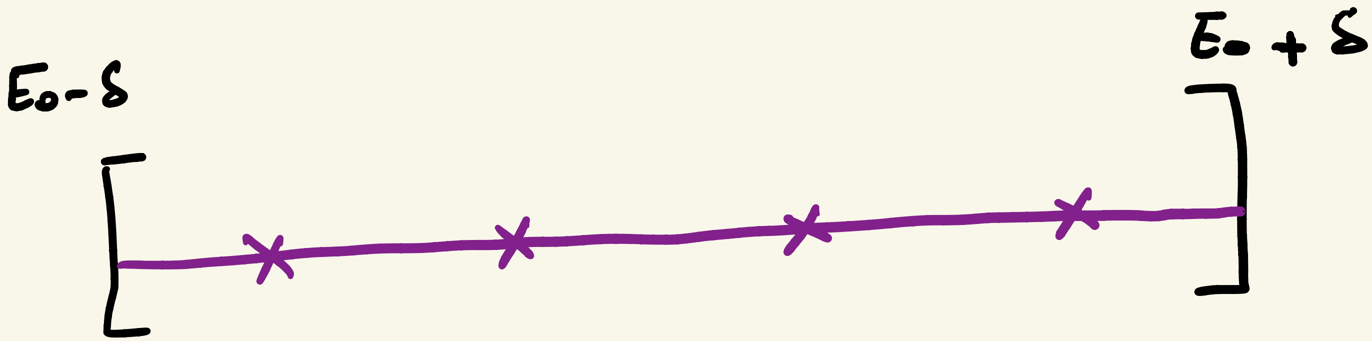


The spectrum of $P_2(h)$ near E_0 is continuous



*) In the trivial case $W \equiv 0$

$$\sigma(P) \cap [E_0 - \delta, E_0 + \delta] = \big(\sigma(P_1) \cup \sigma(P_2)\big) \cap [E_0 - \delta, E_0 + \delta]$$



continuous spectrum with embedded eigenvalues

*) In the general case, we expect that the embedded eigenvalues shift in the lower complex plane as resonances (Fermi's Golden rule)

Definition of resonances Let $\mathcal{R}$ be a small complex neighb. of E_0.

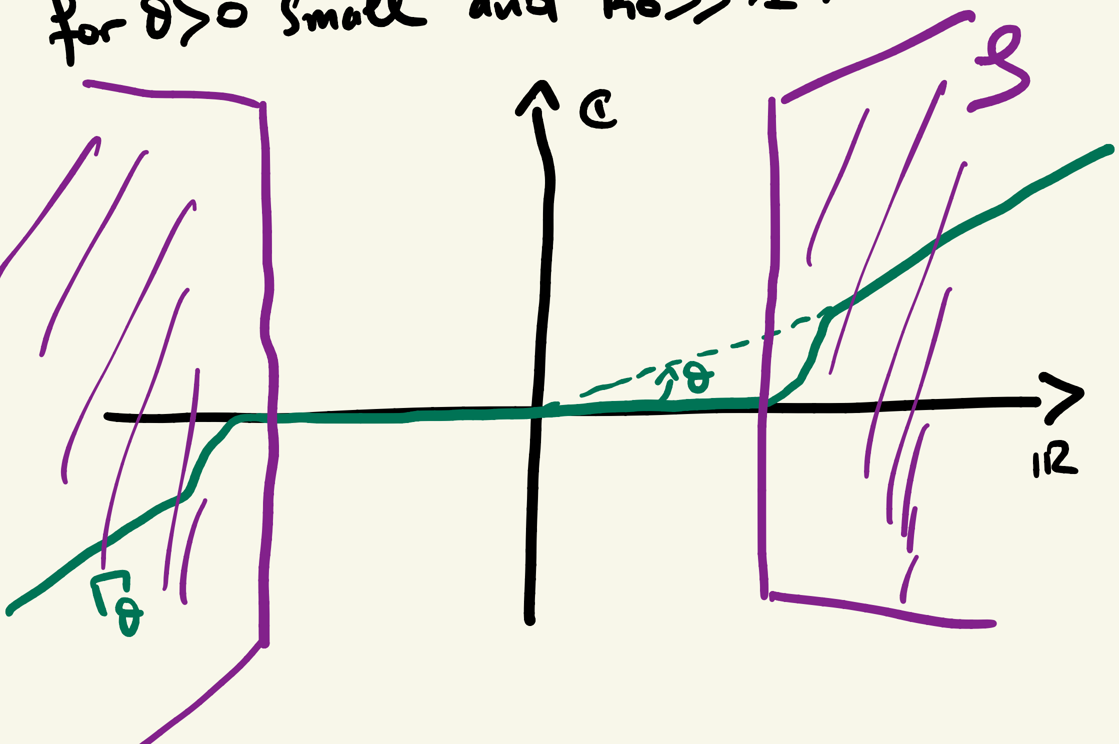
$E \in \mathcal{R}$ is a resonance of P if

$\exists\ 0 \neq w$ outgoing such that $Pw = Ew$.

w outgoing means that $x \longmapsto w(\Psi_\theta(x)) \in L^2(\mathbb{R};\mathbb{C}^2)$

$$C^\infty(\mathbb{R};\mathbb{C}) \ni \Psi_\theta(x) = \begin{cases} x & \text{for } |x| \leq R_0 \\ xe^{i\theta} & \text{for } |x| \geq 2R_0 \end{cases}$$

for $\theta > 0$ small and $R_0 \gg 1$.



Analytic distortion

We need analyticity of V_1, V_2, r_0, r_1 in $\mathcal{S}$

Imaginary part of resonances (resonance width) = Reciprocal of Life time of the quantum particle

$\rightsquigarrow$ Closely related to the interaction between the two classical dynamics of P_1 and P_2.

Intuition: Particles trapped in Γ_1 may change the trajectory to Γ_2 thanks to the interaction and escape to infinity.

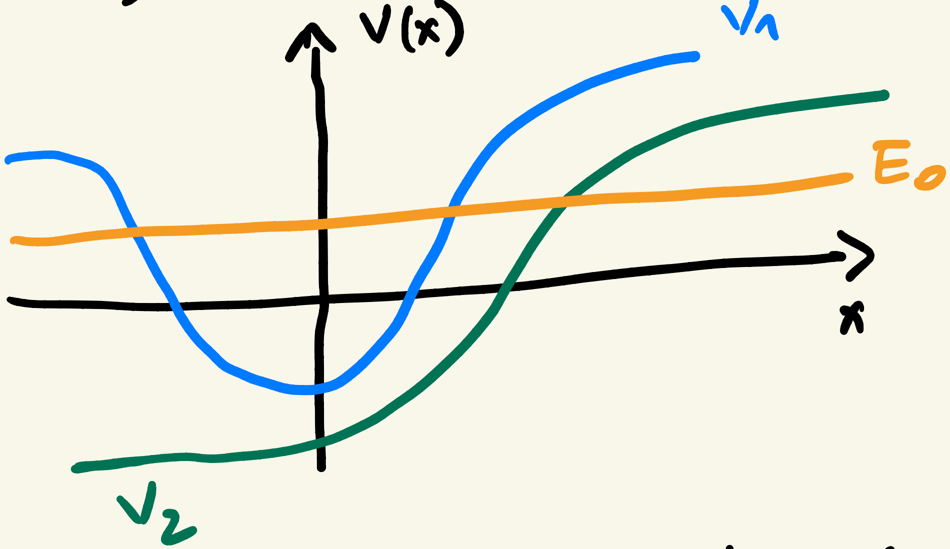
The "probability" of this change of trajectory from Γ_1 to Γ_2 should determine the resonance width.

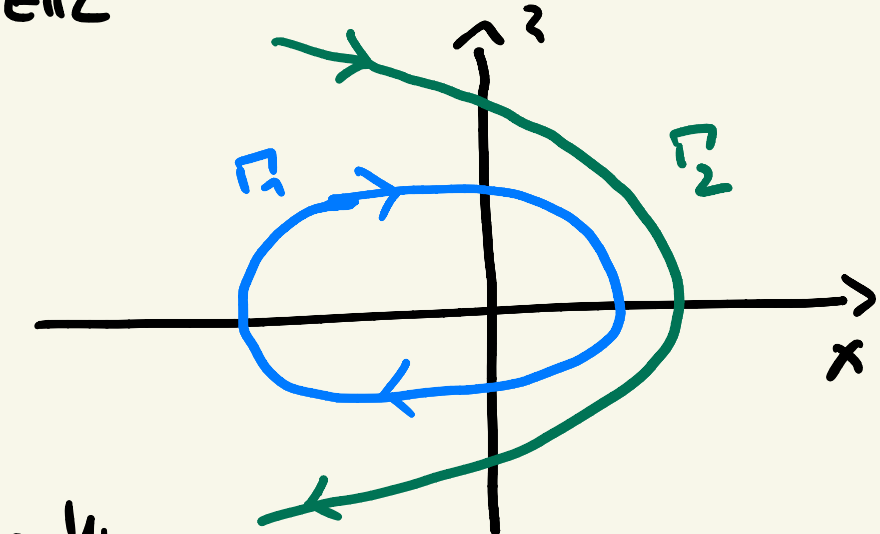
Higher the probability $\Longrightarrow$ wider the resonance width

Case $\Gamma_1 \cap \Gamma_2 = \phi$

This happens when

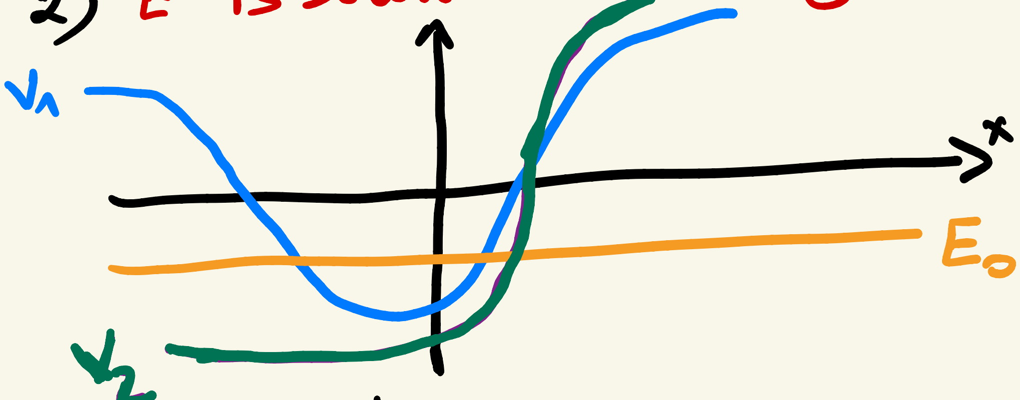
1) V_1 and V_2 don't cross : $\inf_{x\in\mathbb{R}} |V_1(x) - V_2(x)| > 0$

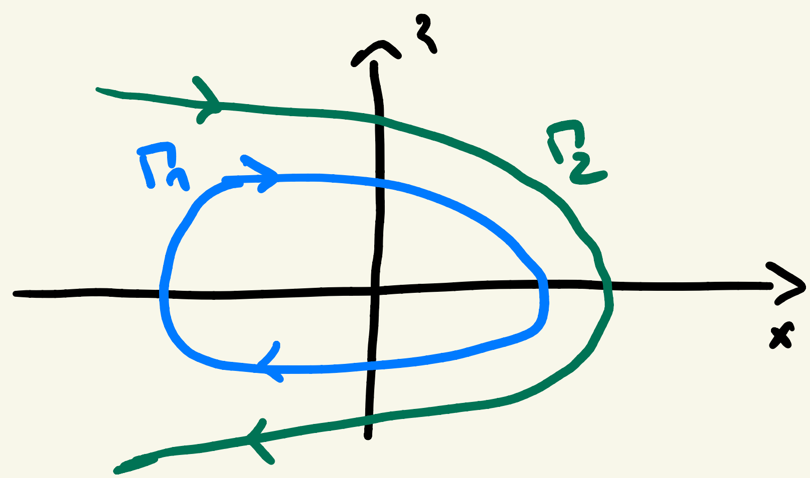




Nakamura '95, Baklouti '96, Grigis-Martinez '14

2) E is below the crossing level :





Ashida '18

In this case :

$$\boxed{\text{Resonance width} = \mathcal{O}(e^{-c/h}),\ c>0 \quad \text{(Phase space tunneling)}}$$

We are interested in the case $\Gamma_1 \cap \Gamma_2 \neq \emptyset$.

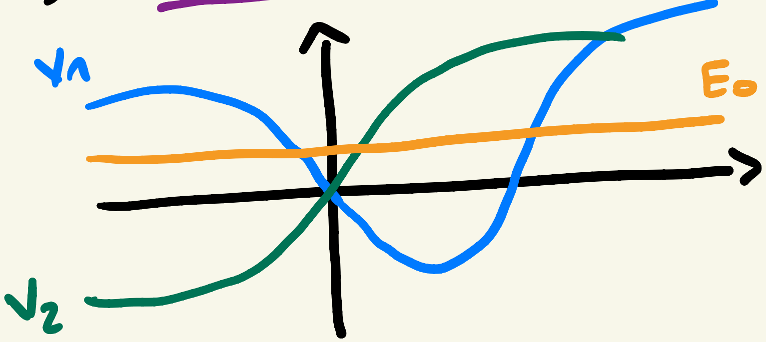
Here we only take a <u>simple model</u>:

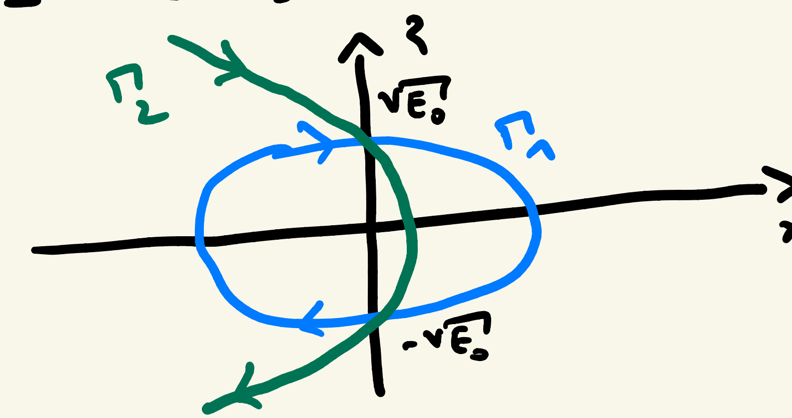
$E_0 \geq 0, \quad V_1(0) = V_2(0) = 0, \quad \{x \in \mathbb{R};\ V_1(x) = V_2(x) \leq E_0\} = \{0\}$

Assume that the contact order $n \in \mathbb{N}$ of V_1 and V_2 at $x=0$ is finite.

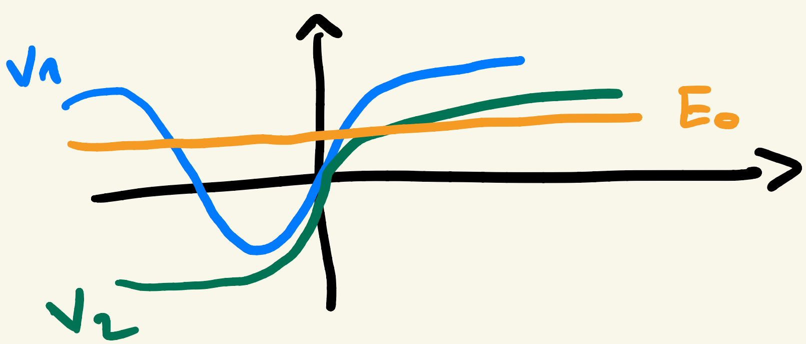
Two cases:

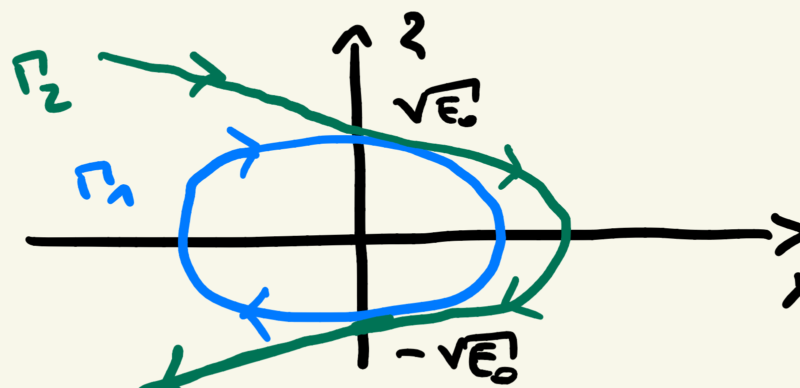
1) $\boxed{E_0 > 0}$ $\Rightarrow \Gamma_1 \cap \Gamma_2 = \{(0, \pm\sqrt{E_0})\}$ with contact order $m = n$





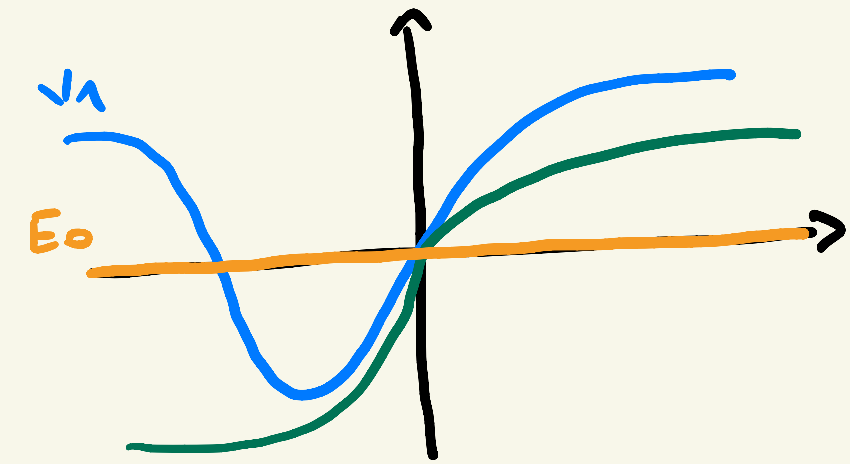
odd n

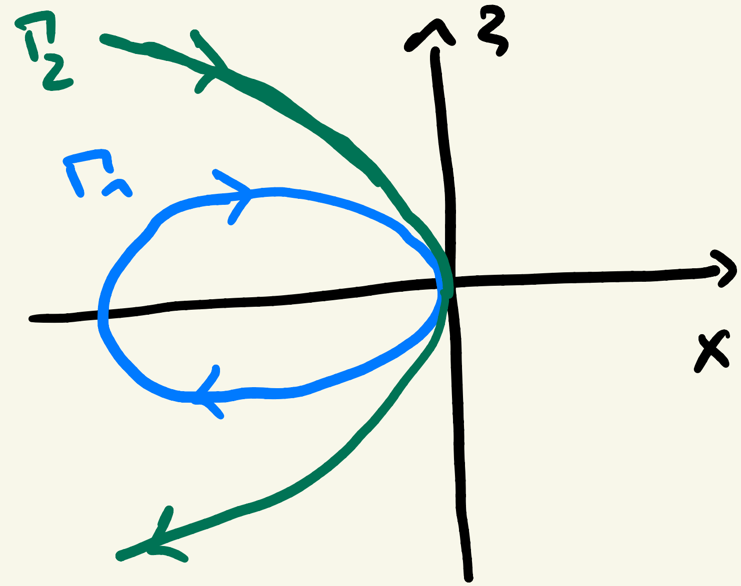




even n

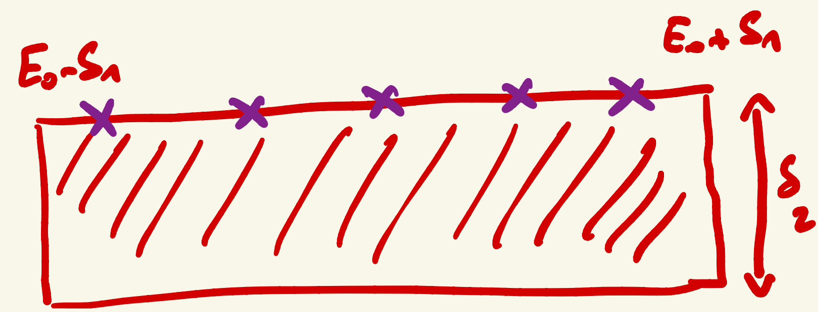
2) $\boxed{E_0=0}$ $\Longrightarrow$ $\Gamma_1 \cap \Gamma_2 = \{(0,0)\}$ with contact order $m=2n$





II/ Main results

Set $R = R(\delta_1,\delta_2) = [E_0-\delta_1, E_0+\delta_1] - i[0,\delta_2]$



$B_h = \{E \in [E_0-\delta_1, E_0+\delta_1] ;\ E \text{ satisfies (BS)}\}$

Scales of δ_1, δ_2 :

$$\delta_1 = \begin{cases} \delta & (E_0>0) \\ L h^{\frac{2}{2n+1}} & (E_0=0) \end{cases} \quad ; \quad \delta_2 = Lh \quad \text{with} \quad \begin{cases} \delta>0 \text{ small } h\text{-independent} \\ L>0 \text{ arbitrarily large} \end{cases}$$

Theorem (A.–Fujiié–Higuchi, '23, '24):

There exists a bijective map

$$z_h : \mathcal{B}_h \longrightarrow \mathrm{Res}(P) \cap \mathbb{R}$$

and a non-negative smooth function $D(E) = D(E,h)$ of E near E_0 uniformly bounded w.r.t. $h>0$ such that for any $E \in \mathcal{B}_h$

$$\boxed{|z_h(E) - E| = \mathcal{O}\left(h^{\frac{m+3}{m+1}}\right)}$$

$$\boxed{\operatorname{Im} z_h(E) = -D(E)\, h^{\frac{m+3}{m+1}} + \mathcal{O}\left(h^{\frac{m+4-\varepsilon}{m+1}}\right)}$$

uniformly as $h \to 0^+$, for some $0 \le \varepsilon < 1$.

The leading term $D(E)$: We assume for simplicity $\boxed{W = r_0(x)}$

1) Case $E_0 > 0$ $(m=n)$

$$D(E) = 2\pi \left(\frac{m+2}{m+1}\right)^2 \left(\frac{2(m+1)!}{|v_n|}\right)^{\frac{2}{m+1}} \frac{E^{-\frac{m}{m+1}}}{A'(E)} r_0(0)^2 R^2 \sin^2\left(\frac{S(E)}{2h} + \vartheta\right)$$

where $\quad v_n = \quad V_2^{(n)}(0) - V_1^{(n)}(0)$

$$R = \begin{cases} 1 & (m \text{ odd}) \\ \cos\left(\frac{\pi}{2(m+1)}\right) & (m \text{ even}) \end{cases} \quad ; \quad \vartheta = \begin{cases} \frac{\pi}{2(m+1)} \operatorname{sgn}(v_n) & (m \text{ odd}) \\ 0 & (m \text{ even}) \end{cases}$$

$S(E)$ is the action of the directed cycle γ

$$S(E) = \int_\gamma \xi \, dx = 2\int_{a(E)}^{0} \sqrt{E - V_1(x)}\, dx + 2\int_0^{c(E)} \sqrt{E - V_2(x)}\, dx$$

2) Case $E_0 = 0$ $(m = 2n)$

$$D(E) = \frac{4\pi^2}{v_0\, q_n^{\frac{2}{2n+1}}\, A'(E)}\, r_0(0)^2\, A_n\left(-\frac{q_n^{\frac{2}{2n+1}}}{v_0}\eta\right)^2$$

$$\eta = \frac{E}{h^{\frac{2}{2n+1}}}$$

A_n is a generalization of the Airy function $(A_1 = Ai)$

$$A_n(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\int_0^{\eta}(\lambda+\tau^2)^n\, d\tau\right) d\eta$$

$$v_0 = \sqrt{V_1'(0)\, V_2'(0)}\,, \qquad q_n = \frac{|v_n|}{n!\,\sqrt{v_0}}\,.$$

III/ Main ideas of the proofs

Microlocal terminologies: Let $\rho_0 \in \mathbb{R}^2$ and $u \in L^2$ with $\|u\|_{L^2} \leq 1$.

$L^2(\mathbb{R};\mathbb{C}^2)$ $=$ L^2

1) We say that u is microlocally 0 at ρ_0 and write $u \equiv 0$ at ρ_0 if $\exists$ a neighborhood Ω of ρ_0 in $\mathbb{R}^2$ such that

$$\|Tu\|_{L^2(\Omega)} = \mathcal{O}(h^\infty)$$

$$Tu(x,\xi,h) = 2^{-1/2}(\pi h)^{-3/4} \int_{\mathbb{R}} e^{i(x-y)\xi/h - (x-y)^2/2h} u(y)\, dy$$

Sjöstrand-FBI transform

2) We say that w is a microlocal solution to $(P-E)w=0$ at $(x_0,\xi_0) \in \mathbb{R}^2$ if $(P-E)w \equiv 0$ at (x_0,ξ_0).

The closed set of points where $u \not\equiv 0$ is the so-called the Frequency Set of u.

<u>Recall that</u>

1) Any locally normalized solution w to $(P-E)w=0$ is microlocally supported on $\Gamma_1(E) \cup \Gamma_2(E)$.

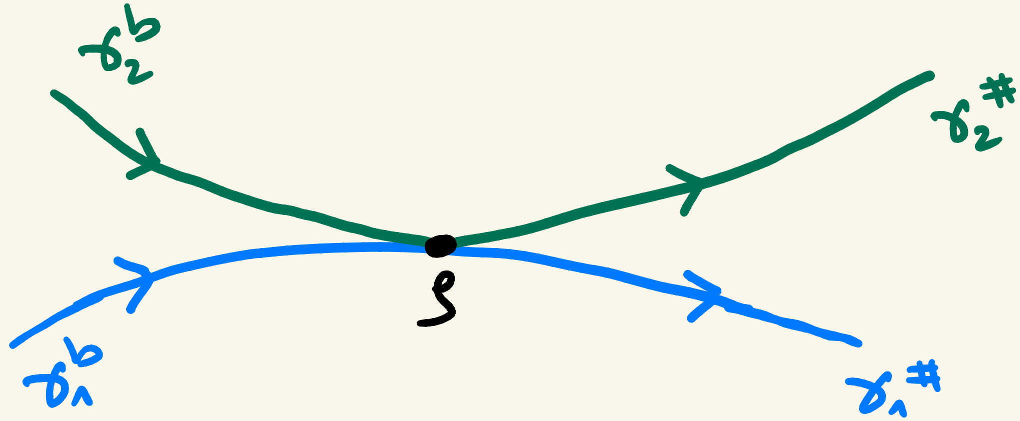
2) If $w \equiv 0$ at $\rho_0 \in \gamma$ a connected component of $(\Gamma_1(E) \cup \Gamma_2(E)) \setminus (\Gamma_1(E) \cap \Gamma_2(E))$ then

$$w \equiv 0 \text{ on } \gamma$$

propagation of singularities

*) The space of microlocal solutions on each connected component of $(\Gamma_1 \cup \Gamma_2) \setminus (\Gamma_1 \cap \Gamma_2)$ is one dimensional.

Let $\rho \in \Gamma_1 \cap \Gamma_2$ be a crossing point and Ω a small neighb. of ρ.



We construct 4 WKB solutions $f_1^b, f_1^\#, f_2^b, f_2^\#$ to the system $(P-E)w \equiv 0$ on $\gamma_1^b, \gamma_1^\#, \gamma_2^b, \gamma_2^\#$.

$$f_j^\bullet = e^{i\varphi_j^\bullet(x)/h}\, \sigma_j^\bullet(x,h) \quad , \quad \boxed{\varphi_j^\bullet(x) = \pm \int_0^x \sqrt{E - V_j(t)}\, dt}$$

$\pm$ corresponds to if $\gamma_j^\bullet$ is in $\{\pm\xi > 0\}$.

Theorem (A.-Fujiié-Higuchi '23, '24)

For $E \in \mathbb{R}$ and $h > 0$ small, $\exists$ a 2×2 matrix $T = T(E,h)$ such that if a locally normalized solution w satisfies

$$(P-E)w \equiv 0 \quad \text{on } \Omega$$

$$w \equiv \alpha_j f_j \quad \text{on } \gamma_j \qquad (j = 1, 2, \ldots = \sharp, b)$$

then

$$\begin{pmatrix} \alpha_1^\sharp \\ \alpha_2^\sharp \end{pmatrix} = T(E,h) \begin{pmatrix} \alpha_1^b \\ \alpha_2^b \end{pmatrix}$$

$$T = \mathrm{Id} - i h^{\frac{1}{m+1}} \begin{pmatrix} 0 & \overline{w} \\ w & 0 \end{pmatrix} + \mathcal{O}\left(h^{\frac{2-\varepsilon}{m+1}}\right), \quad (E_0 > 0, \rho = \rho_+)$$

$$T = \mathrm{Id} - i h^{\frac{1}{m+1}} \begin{pmatrix} 0 & w \\ \overline{w} & 0 \end{pmatrix} + \mathcal{O}\left(h^{\frac{2-\varepsilon}{m+1}}\right), \quad (E_0 > 0, \rho = \rho_-)$$

$$iT = \mathrm{Id} - i h^{\frac{1}{m+1}} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} + \mathcal{O}\left(h^{\frac{2-\varepsilon}{m+1}}\right), \quad (E_0 = 0, \rho = (0,0))$$

where $w, k \in \mathbb{C}$ given explicitly.

The proof of the previous Theorem is based on

1) Construction of suitable local solutions to the system $(P\_E)\,w = 0$

2) Study of oscillatory integrals with degenerate phase

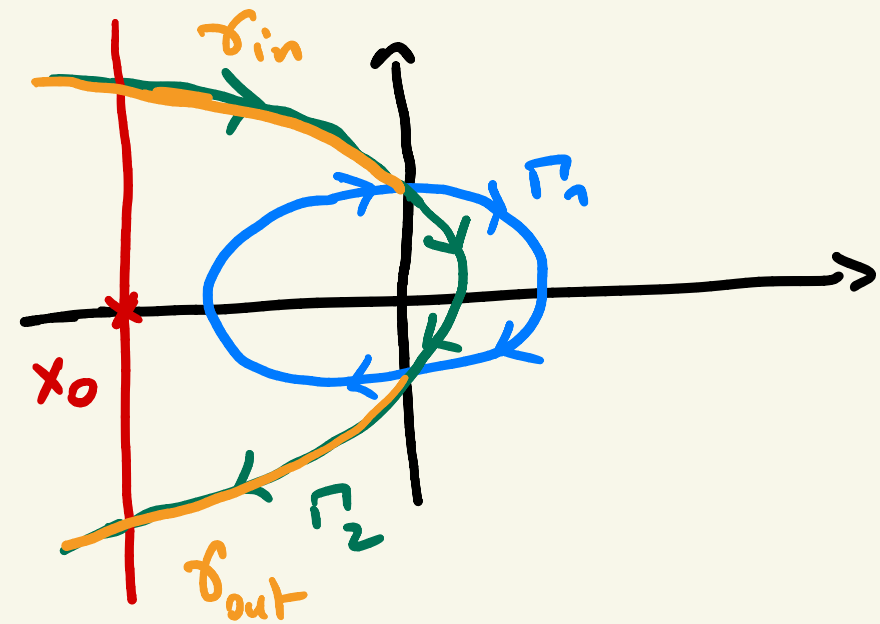
Degenerate stationary phase

From microlocal connection formulae to resonances

Suppose $E \in \mathrm{Res}(P) \cap \mathbb{R}$ and w a corresponding resonant state: $Pw = Ew$.

$\Rightarrow \; w \in L^2([x_0, +\infty))$, $\forall x_0 \in \mathbb{R}$.

Take $x_0 < a_0$.



Writing $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, we have

$$0 = \langle (P - E) w, w \rangle_{L^2([x_0, +\infty))}$$

$$= \| h D_x w \|^2_{L^2} - E \| w \|^2_{L^2} - h^2 \langle w'(x_0), w(x_0) \rangle_{L^2} + 2h \,\mathrm{Re} \langle r_0 w_2, w_1 \rangle_{L^2}$$

This gives us

$$\operatorname{Im} E = -\frac{h^2}{\|w\|^2_{L^2([x_0,+\infty))}} \left(w_1'(x_0)\,\overline{w_1(x_0)} + w_2'(x_0)\,\overline{w_2(x_0)} \right)$$

Near $x = x_0$, we are far from the crossing and turning points

So the WKB expression is valid:

Modulo $\mathcal{O}(h)$ we have $\quad w_1 = 0 \quad$ and

$$w_2 \sim \frac{\alpha_{out}}{(E - V_2(x))^{1/4}} \exp\left(-\frac{i}{h}\int_0^x \sqrt{E - V_2(t)}\, dt\right)$$

$$w_2' \sim \frac{i}{h}\, \overline{\alpha_{out}}\, (E - V_2(x))^{1/4} \exp\left(\frac{i}{h}\int_0^x \sqrt{E - V_2(t)}\, dt\right)$$

for some constant $\alpha_{out} \in \mathbb{C}$ which yields

$$\boxed{w_2'(x_0)\,\overline{w_2(x_0)} \sim \frac{i}{h}\,|\alpha_{out}|^2}$$

The microlocal behavior of w near x_0 is

$$w \equiv \alpha_{out} f_{out} \qquad \text{on } \gamma_{out}$$

$$w \equiv 0 \qquad \text{on } \gamma_{in}$$

If w is normalized on any portion γ_1 of Γ_1 free from crossing and turning points then

$$w \equiv f_{\gamma_1} \quad \text{on } \gamma_1$$

and $$\|w\|^2_{L^2([x_0,+\infty))} = 2A'(E) + \mathcal{O}\left(h^{1/3} + h^{\frac{1}{m+1}}\right).$$

We finally obtain

$$\operatorname{Im} E = -\frac{h\,|\alpha_{out}|^2}{2A'(E)} + \mathcal{O}\left(h^{4/3} + h^{\frac{m+2}{m+1}}\right).$$

Gracias!