Quantum Double Models:

From topological materials to quantum computation.

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Motivation

“God is a mathematician of a very high order and He used advanced mathematics in constructing the universe.”
- Paul Dirac

”With the utmost respect Mr. Dirac, are you sure?”
- J. Lorca Espiro

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The simplest of them all: The Toric Code, generalities

Spin $1/2$ model on the square lattice (periodic boundary conditions).

Hamiltonian:

$$H = -\sum_v A_v - \sum_p B_p$$

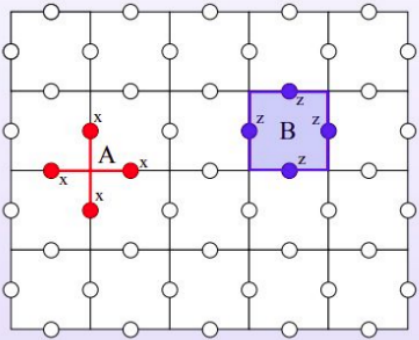
The Hamiltonian is the sum of two kind of terms (stabilizers):

$$A_v = \prod_{i \in v} \sigma_{x,i}\,, \qquad B_p = \prod_{i \in p} \sigma_{z,i}$$

All these terms commute:

$$[A_i, A_j] = [B_i, B_j] = [A_i, B_j] = 0$$

Spins sit on the **edges**:



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$$H = -\sum A_v - \sum B_p$$

Since all the stabilizers A and B commute, a GS is identified by:

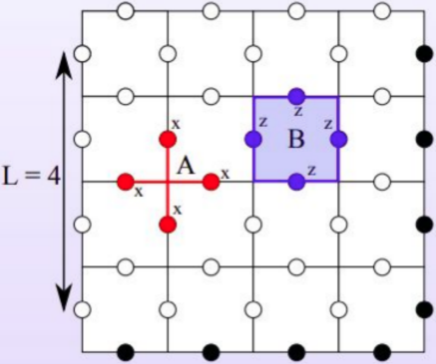
$$A_v = \prod \sigma_{x,i} = 1\,, \qquad B_p = \prod \sigma_{z,i} = 1.$$

- Number of physical spins: $N = 2L^2$
- Number of stabilizers: $N_A = L^2$, $N_B = L^2$
- 2 constraints:

$$\prod A_v = 1\,, \quad \prod B_p = 1.$$

- Number of ground states:

$$2^{N-(N_A+N_B-2)} = 4.$$



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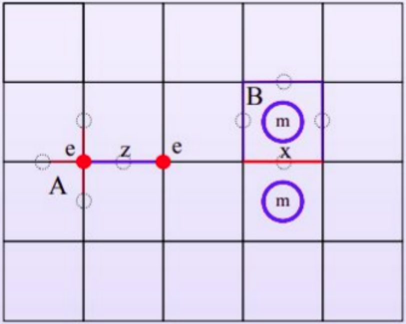
If $A_v = -1$ or $B_p = -1$, a localized excitation appears with energy 2.

- $A = -1$: electric charge e.
- $B = -1$: magnetic vortex m.

Local operators σ_z or σ_x create pairs of excitations:



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• Trivial symmetry (stabilizers): Product of stabilizers A_v or B_p, operates as the identity over the ground states.

• Non-trivial symmetry (not a product of stabilizers): String with non-trivial homology and not fixed value.

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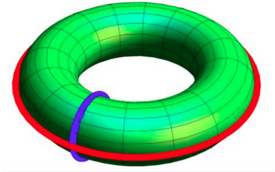

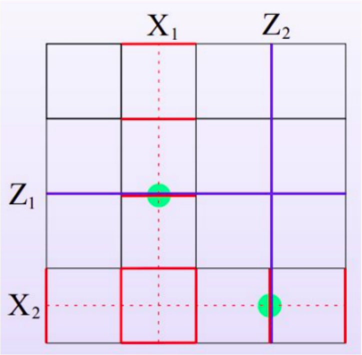

Quantum Doubles Models and Gauge Theory picture

(a) a discretized manifold L.

$$\{E\} \to G \quad \Rightarrow \quad \mathcal{H} = \bigoplus_{\{E\}} \mathbb{C}\,[G]_e \quad ,$$

$$H := \sum_{v\in L} (\mathbb{1}_v - A_v) + \sum_{p\in L} (\mathbb{1}_p - B_p) \quad ,$$

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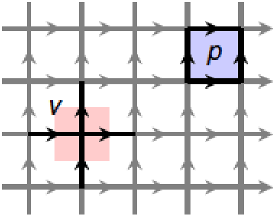

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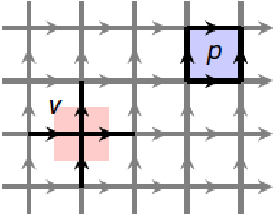
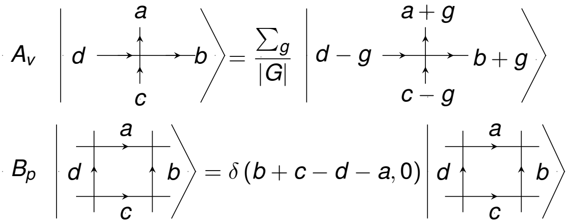
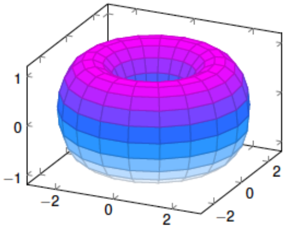
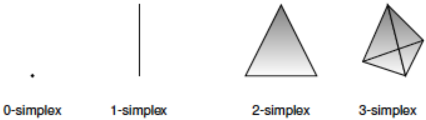




Mathematical Structure

Discretization of Manifolds





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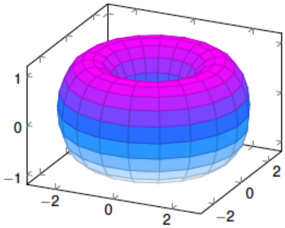

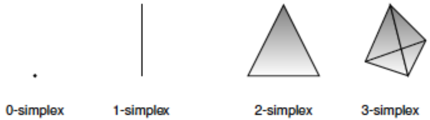

Configuration

$$\{V\} \sim C_0 \to G_0$$

$$\{E\} \sim C_1 \to G_1$$

$$\{F\} \sim C_2 \to G_2$$

$$\vdots$$

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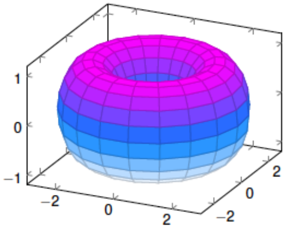

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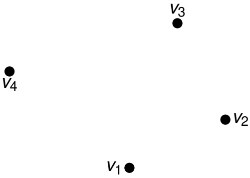

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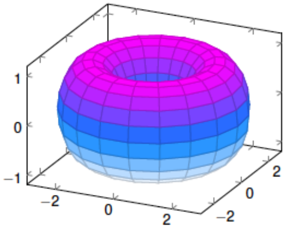

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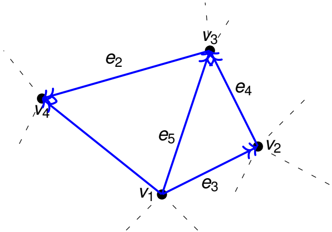

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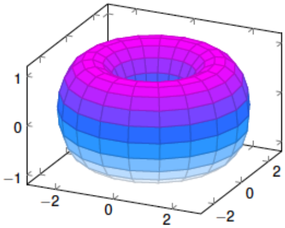

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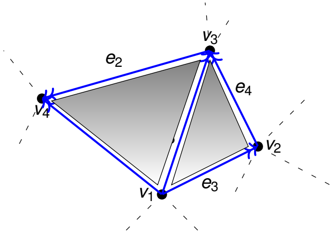

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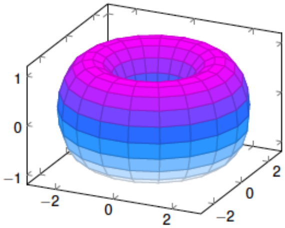

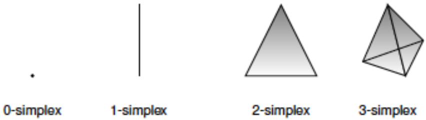

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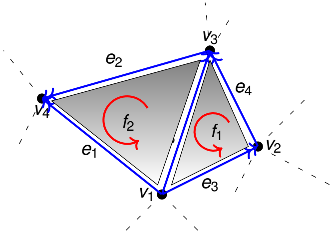

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For all $p \in \mathbb{Z}$, let $\hom(C, G)^p := \prod_n \mathrm{Hom}(C_n, G_{n-p})$. The components of $f \in \hom(C, G)^p$ are denoted $f_n : C_n \to G_{n-p}$.

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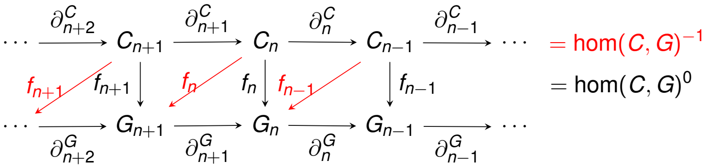

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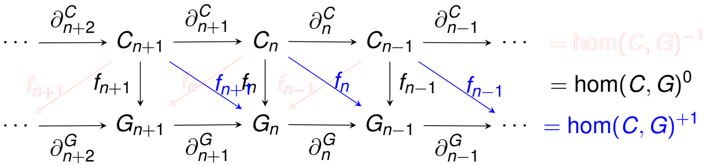

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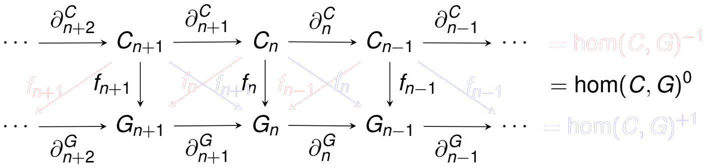

The $(\hom(C,G)^\bullet, \delta^\bullet)$ cochain complex

Definition

We define the group homomorphism $\delta^p : \hom(C,G)^p \to \hom(C,G)^{p+1}$

$$(\delta^p f)_n = f_{n-1} \circ \partial_n^C - (-1)^p \partial_{n-p}^G \circ f_n \quad .$$

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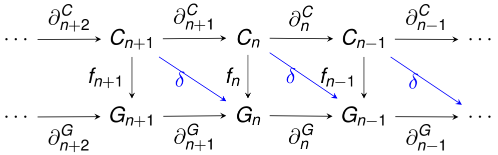

$= \hom(C,G)^0$

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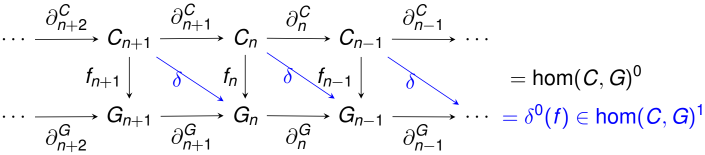

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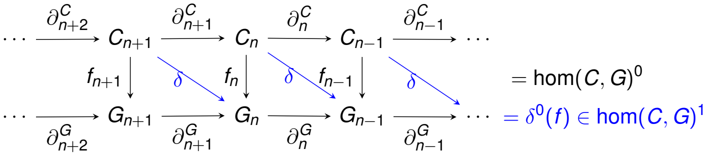

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Definition (Cohomology)

Cohomology groups with coeff. in the chain complex

$$H^n(C,G) := \ker(\delta^n)/\mathrm{im}(\delta^{n-1})$$

The Models

Configuration and Representation Hilbert spaces

Let $f \in \hom(C, G)^0$, we construct the states $|f\rangle = \bigotimes_{n,x \in K_n} |f_n(x)\rangle$,

$$\mathcal{H} \simeq \overline{\text{span}_{\forall f}\{|f\rangle\}} \simeq \bigotimes_{n,x \in K_n} \mathbb{C}[G_n] \quad \text{and} \quad \dim(\mathcal{H}) < \infty \quad .$$

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Definition (p-characters and dualization)

We take $\chi_{\hat{\pi}}(f) \simeq \langle \hat{\pi} | f \rangle := \prod_{n, x \in K_n} \langle \hat{\pi}_n | f_n \rangle_x \sim e^{i \sum_{n,x} \pi_n(f_n)_x}$

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This defines $\hat{\pi} \in \text{hom}(C, G)_0$ (dual space) with $|\hat{\pi}\rangle = \bigotimes_{n,x \in K_n} |\hat{\pi}_n(x)\rangle$,

$$\hat{\mathcal{H}} \simeq \overline{\text{span}_{\forall \hat{\pi}}\{|\pi\rangle\}} \simeq \bigotimes_{n,x \in K_n} \mathbb{C}\left[\hat{G}_n\right] \quad \text{and} \quad \dim(\hat{\mathcal{H}}) < \infty \quad .$$

Global and Local operators

For all $t \in \text{hom}^{-1}$, $\hat{\pi} \in \text{hom}_1$ it follows that $\langle \hat{\pi} | \delta^0 \circ \delta^{-1} t \rangle = 1$. Hence,

Operators	On $\|f\rangle \in \mathcal{H}$	On $\|\hat{\rho}\rangle \in \hat{\mathcal{H}}$
Shift	$P^{\delta^{-1}t} \|f\rangle := \|f + \delta^{-1} t\rangle$	$P^{\delta^{-1}t} \|\hat{\rho}\rangle = \langle \hat{\rho} \| \delta^{-1} t\rangle \|\hat{\rho}\rangle$
Clock	$Q_{\delta_1 \hat{\pi}} \|f\rangle := \langle \delta_1 \hat{\pi} \| f \rangle \|f\rangle$	$Q_{\delta_1 \hat{\pi}} \|\hat{\rho}\rangle = \|\hat{\rho} + \delta_1 \hat{\pi}\rangle$

$$Q_{\delta_1 \hat{\pi}} P^{\delta^{-1} t} = \langle \hat{\pi} | \delta^0 \circ \delta^{-1} t \rangle P^{\delta^{-1} t} Q_{\delta_1 \hat{\pi}} \quad \text{they commute!!!}$$

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Clock	$Q_{\delta_1\hat{\pi}}\|f\rangle := \langle\delta_1\hat{\pi}\|f\rangle\|f\rangle$	$Q_{\delta_1\hat{\pi}}\|\hat{\rho}\rangle = \|\hat{\rho} + \delta_1\hat{\pi}\rangle$

$$Q_{\delta_1\hat{\pi}}P^{\delta^{-1}t} = \langle\hat{\pi}|\delta^0 \circ \delta^{-1}t\rangle P^{\delta^{-1}t}Q_{\delta_1\hat{\pi}} \quad \text{they commute!!!}$$

Projector Operators (global)
$\mathcal{A}_{\hat{\pi}} := \frac{\sum_t \langle\hat{\pi}\|t\rangle P^{\delta^{-1}t}}{\|\text{hom}^{-1}\|} \quad , \quad \mathcal{B}^t := \frac{\sum_{\hat{\pi}} \langle t\|\hat{\pi}\rangle Q_{\delta_1\hat{\pi}}}{\|\text{hom}_1\|} \quad , \quad \Pi^t_{\hat{\pi}} = \mathcal{A}_{\hat{\pi}}\mathcal{B}^t$

Global and Local operators

For all $t \in \text{hom}^{-1}$, $\hat{\pi} \in \text{hom}_1$ it follows that $\langle \hat{\pi} | \delta^0 \circ \delta^{-1} t \rangle = 1$. Hence,

Operators	On $\|f\rangle \in \mathcal{H}$	On $\|\hat{\rho}\rangle \in \hat{\mathcal{H}}$
Shift Clock	$P^{\delta^{-1}t} \|f\rangle := \|f + \delta^{-1} t\rangle$ $Q_{\delta_1 \hat{\pi}} \|f\rangle := \langle \delta_1 \hat{\pi} \| f \rangle \|f\rangle$	$P^{\delta^{-1}t} \|\hat{\rho}\rangle = \langle \hat{\rho} \| \delta^{-1} t \rangle \|\hat{\rho}\rangle$ $Q_{\delta_1 \hat{\pi}} \|\hat{\rho}\rangle = \|\hat{\rho} + \delta_1 \hat{\pi}\rangle$

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If $x \in K_n$ and $g \in G_{n-1}$, $\hat{r} \in \hat{G}_{n+1}$, then

for	*Proj. op. (locally compact)*
$gx^* \in \text{hom}^{-1}\ ,\ \hat{r} x_* \in \text{hom}_1$	$\mathcal{A}_{\hat{r} x_*} := A_{\hat{r}_x}\ ,\ \mathcal{B}^{gx^*} := B^{g_x}$

"Gauge" (and "Holonomy") Equivalence

Proposition (Gauge equivalence (homotopy))

Let $g \in hom^0$ *and* $t \in hom^{-1}$. *Let* $|f\rangle = P^{\delta^{-1}t}|g\rangle = |g + \delta^{-1}t\rangle$ *then*

$$\mathcal{A}_{\hat{0}}|f\rangle = \mathcal{A}_{\hat{0}}|g\rangle \quad \text{and we write } f \simeq g \quad \text{(equivalence relation!)}$$

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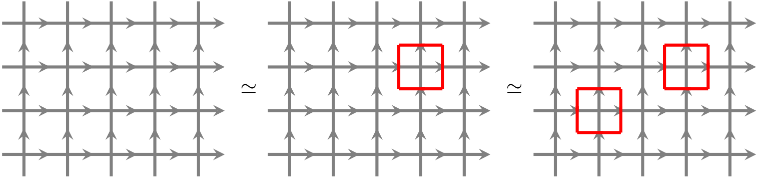

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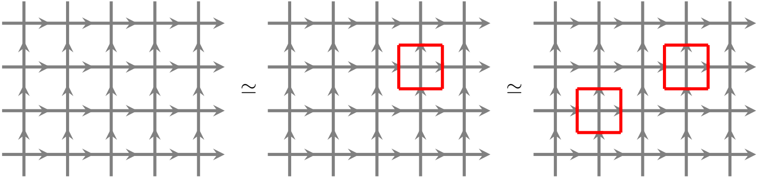
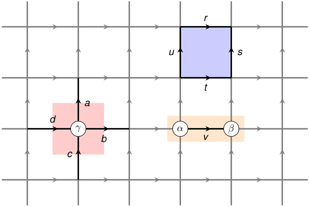


Proposition (Holonomy equivalence (co-homotopy))

Let $\hat{\rho} \in hom_0$ *and* $\hat{\pi} \in hom_1$. *Let* $|\hat{\omega}\rangle = Q_{\delta_1\hat{\pi}}|\hat{\rho}\rangle = |\hat{\rho} + \delta_1\hat{\pi}\rangle$ *then*

$$\mathcal{B}^0|\hat{\omega}\rangle = \mathcal{B}^0|\hat{\rho}\rangle \quad \textit{and we write} \quad \hat{\omega} \simeq \hat{\rho} \quad \textit{(equivalence relation!)}$$

Operators example: $0, 1$ gauge



$$
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial_4^C} & C_3 & \xrightarrow{\partial_3^C} & C_2 & \xrightarrow{\partial_2^C} & C_1 & \xrightarrow{\partial_1^C} & C_0 & \xrightarrow{\partial_0^C} & 0 \\
 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \\
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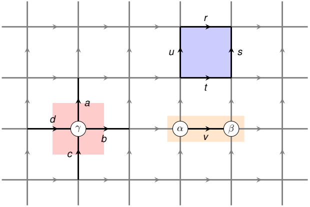
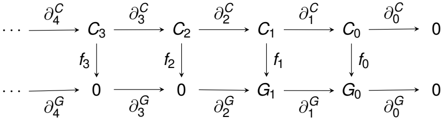
$$





Let $\alpha, \beta, \gamma \in G_0$, $\hat{\mu} \in \hat{G}_0$, $a, b, c, d, r, s, t, u, g \in G_1$ and $\hat{h} \in \hat{G}_1$ then

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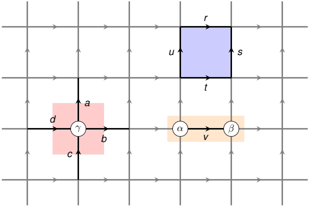

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$$A_{\hat{0}_x} = \frac{\sum_g P^{\delta^{-1} g \otimes x^*}}{|G_1|}$$

$$A_{\hat{0}_v} \left| \begin{array}{c} a \\ \uparrow \\ d \rightarrow \gamma \rightarrow b \\ \uparrow \\ c \end{array} \right\rangle := \frac{\sum_g \partial_1^G(g) \triangleright}{|G_1|} \left| \begin{array}{c} g+a \\ \uparrow \\ d-g \rightarrow \gamma \rightarrow g+b \\ \uparrow \\ c-g \end{array} \right\rangle$$

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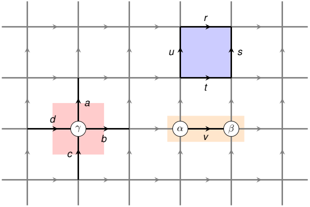

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$$B^{0_x} = \begin{cases} \dfrac{\sum_{\hat{h}} Q_{\delta_1 \hat{h} p_*}}{|\hat{G}_1|} \\ \dfrac{\sum_{\hat{\mu}} Q_{\delta_1 \hat{\mu} l_*}}{|\hat{G}_0|} \end{cases}$$

$$B^{0_p}\left| u, r, s, t \right\rangle = \frac{\sum_{\hat{h}} \left\langle \hat{h} | s+t-u-r \right\rangle_p}{|\hat{G}_1|}\left| u, r, s, t \right\rangle$$

$$B^{0_l}\left| \alpha \xrightarrow{g} \beta \right\rangle = \frac{\sum_{\hat{\mu}} \left\langle \hat{\mu} | \partial_1^G(g) + \alpha - \beta \right\rangle_l}{|\hat{G}_0|}\left| \alpha \xrightarrow{g} \beta \right\rangle$$

Dynamics and Evolution

Definition (Hamiltonian (à la Kitaev))

We define the Hamiltonian operator $H : \mathcal{H} \to \mathcal{H}$ as:

$$H := -\ln\left(\Pi_{\hat{0}}^{0}\right) = \ln(2)\left(\sum_{n,x\in K_n}\left(\mathbb{1}_x - A_{\hat{0}_x}\right) + \sum_{n,y\in K_n}\left(\mathbb{1}_x - B^{0_y}\right)\right).$$

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The generator of the dynamics is $\delta(O) := [H, O]$ for any $O \in \mathcal{O}$, then

$$\mathcal{U}_t(O) := e^{it\delta}(O) \quad \text{gives the time evolution for } O \quad .$$

Proposition

It follows that $\delta(A_{\hat{\pi}}) = \delta\left(B^t\right) = 0\ \forall\, \hat{\pi} \in hom_0$ *and* $t \in hom^0$ *. Thus,* $\mathcal{U}_t(A_{\hat{\rho}}) = \mathcal{U}_t(B^{\omega}) = \mathbb{1}_{\mathcal{H}}$ *, i.e. they are time independent .*

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Notice that the $Z(X) = \mathrm{Tr}\left(e^{-\beta H}\right) = \mathrm{Tr}\left(\left(\Pi^{0}_{\hat{0}}\right)^{\beta}\right) = GSD^{\beta}$ is a TP.

Ground States Characterization

Ground States

Proposition (Projector into the GS)

$A\, |\Psi\rangle \in \mathcal{H}_0$ (GSS) iff $\mathcal{A}_{\hat{0}}\, |\Psi\rangle = |\Psi\rangle$ and $\mathcal{B}^{\hat{0}}\, |\Psi\rangle = |\Psi\rangle$. Then $\Pi^0_{\hat{0}} = \mathcal{A}_{\hat{0}}\mathcal{B}^0$ is a projector into $\mathcal{H}_0$. This is GSD $= Tr\left(\Pi^0_{\hat{0}}\right)$.

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Proposition (Frustration Free Models and Seed states)

- *By construction $|0_{\hat{0}}\rangle := \mathcal{A}_{\hat{0}}\,|0\rangle \in \mathcal{H}_0$ in the configuration basis.*

Then, from the seed state above $P^f\,|0_{\hat{0}}\rangle := |f_{\hat{0}}\rangle \in \mathcal{H}_0$ iff $f \in \ker\left(\delta^0\right)$.

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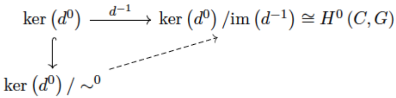

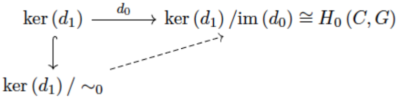

Ground State Degeneracy Theorem

Theorem (Dimension of the ground state subspace!!!)

The dimension of the ground state subspace $\mathcal{H}_0$ is given by:

$$GSD = |H^0(C, G)| \quad \text{or equivalently} \quad GSD = |H_0(C, G)| \quad ,$$

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The first characterization gives us a way to calculate the GSD for general manifolds by means of the *Universal Coefficient Theorem*

$$\left| H^0(C, G) \right| \cong \prod_n |H^n(C, H_n(G))| \qquad \text{where}$$

$$H^n(C, H_n(G)) \cong \mathrm{Hom}(H_n(C), H_n(G)) \oplus \mathrm{Ext}(H_{n-1}(C), H_n(G)) \qquad .$$

GSD Calculation Examples

Ground State Degeneracy of the Toric Code

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\partial_3^C} & C_2 & \xrightarrow{\partial_2^C} & C_1 & \xrightarrow{\partial_1^C} & C_0 & \xrightarrow{\partial_0^C} & 0 \\
 & & \Big\downarrow f_2 & & \Big\downarrow f_1 & & \Big\downarrow f_0 & & \\
0 & \xrightarrow[\partial_3^G]{} & 0 & \xrightarrow[\partial_2^G]{} & G_1 & \xrightarrow[\partial_1^G]{} & 0 & \xrightarrow[\partial_0^G]{} & 0
\end{array} = \hom(C, G)^0$$

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\end{array}
\qquad = \hom(C, G)^0
$$

Let $C = C(T^2)$ and $G_1 = \mathbb{Z}_2$. Then $H_{\text{QDM}} = -\sum_{x \in K_0} A_{\hat{0}_x} - \sum_{y \in K_2} B^{0_y}$.

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Homology	*n*
$H_n\left(T^2\right)$	$\mathbb{Z}$ $\mathbb{Z} \oplus \mathbb{Z}$ $\mathbb{Z}$

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Homology	*n*
$H_n(G)$	0 $\mathbb{Z}_2$ 0

Hence, $\text{GSD} = |H^0(C, G)| = |H^1(C, H_1(G))| = 2^2$.

GSD $\mathbb{Z}_2$, $\mathbb{Z}_4$ Abelian $1,2$-gauge theory over S^2

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\partial_3^C} & C_2 & \xrightarrow{\partial_2^C} & C_1 & \xrightarrow{\partial_1^C} & C_0 & \xrightarrow{\partial_0^C} & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\
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Let $C = C(S^2)$, $G_1 = \mathbb{Z}_2 = \{1, -1\}$, $G_2 = \mathbb{Z}_4 = \{1, i, -1, -i\}$, with $\partial_2^G(i) = -1$.

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Homology	n
$H_n\left(S^2\right)$	0
	0
	$\mathbb{Z}$

GSD $\mathbb{Z}_2$, $\mathbb{Z}_4$ Abelian $1, 2$-gauge theory over S^2

$$\begin{array}{ccccccccc}
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Homology	*n*
	0
$H_n\left(S^2\right)$	0
	$\mathbb{Z}$

Homology	*n*
	0
$H_n(G)$	0
	$\mathbb{Z}_2$

GSD $\mathbb{Z}_2, \mathbb{Z}_4$ Abelian $1, 2$-gauge theory over S^2

$$
\begin{array}{ccccccccc}
0 & \xrightarrow{\partial_3^C} & C_2 & \xrightarrow{\partial_2^C} & C_1 & \xrightarrow{\partial_1^C} & C_0 & \xrightarrow{\partial_0^C} & 0 \\
 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \\
0 & \xrightarrow[\partial_3^G]{} & \mathbb{Z}_4 & \xrightarrow[\partial_2^G]{} & \mathbb{Z}_2 & \xrightarrow[\partial_1^G]{} & 0 & \xrightarrow[\partial_0^G]{} & 0
\end{array}
$$

Let $C = C(S^2)$, $G_1 = \mathbb{Z}_2 = \{1, -1\}$, $G_2 = \mathbb{Z}_4 = \{1, i, -1, -i\}$, with $\partial_2^G(i) = -1$.

Homology	n
	0
$H_n\left(S^2\right)$	0
	$\mathbb{Z}$

Homology	n
	0
$H_n(G)$	0
	$\mathbb{Z}_2$

$$\text{GSD} = |H^0(C, G)| = |H^2(C, H_2(G))| = |\text{Hom}(H_2(C), H_2(G))| = 2.$$

Final remarks and Future work

(We are finishing, at last...)

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• This opens a huge class of new materials with exotic statistics and fault tolerant error correction codes in more dimensions.

Future Work

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Thank you

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