Transfer matrix methods for m-channel unitary operators

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Plan of the talk

1. Background: The Hermitian case, transfer matrices and spectral averaging formula
2. Unitary m-channel operators and spectral averaging formula
3. Proof ideas and future directions (very brief)

The Hermitian case: Transfer matrices and spectral averaging formula

- 1d Schrödinger or Jacobi operator on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$

$$(H\psi)_n = -\psi_{n+1} + v_n \psi_n - \psi_{n-1}$$

  where $v_n \in \mathbb{R}$.

- $H\psi = z\psi$ leads to

$$\psi_{n+1} = (v_n - z)\,\psi_n - \psi_{n-1}$$

  or

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \underbrace{\begin{pmatrix} v_n - z & -1 \\ 1 & 0 \end{pmatrix}}_{=: T_n^z} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}$$

- $\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = T_n^z \, T_{n-1}^z \cdots T_0^z \begin{pmatrix} \psi_0 \\ \psi_{-1} \end{pmatrix}$.

Theorem (Carmona-Lacroix)

Let H be the operator on the half-line and let μ be the spectral measure at $|0\rangle \in \ell^2(\mathbb{Z}_+)$, and let $f \in C_b(\mathbb{R})$, then

$$\langle 0|f(H)|0\rangle = \int_{\mathbb{R}} f(E)\mathrm{d}\mu(E) = \lim_{n\to\infty} \int_{\mathbb{R}} \frac{f(E)\,\mathrm{d}E}{\pi \left\| T_n^E \cdots T_1^E T_0^E \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2}$$

Corollary (Last-Simon)

Assume for $p > 1$ and $a < b$ that there are $u_{n,E}$ such that

$$\liminf_{n\to\infty} \int_a^b \left\| T_n^E \cdots T_1^E T_0^E \begin{pmatrix} u_{n,E} \\ 1 \end{pmatrix} \right\|^{2p} \mathrm{d}E < \infty$$

then the spectrum of H is purely absolutely continuous in (a,b) and has a $L^p(a,b)$ density w.r.t. the Lebesgue measure.

Hermitian m-channel operators

- Consider a graph $\mathbb{G}$ partitioned into finite shells S_n , $|S_n| \geq m$

$$\ell^2(\mathbb{G}) = \bigoplus_{n=0}^{\infty} \ell^2(S_n) = \bigoplus_{n=0}^{\infty} \mathbb{C}^{S_n}$$

$$\psi = \bigoplus_{n=0}^{\infty} \psi_n , \qquad \psi_n \in \ell^2(S_n) = \mathbb{C}^{S_n}$$

- We say that H is an m-channel operator across this partition if H can be written as

$$(H\psi)_n = -\Phi_n \Upsilon^*_{n+1} \psi_{n+1} - \Upsilon_n \Phi^*_{n-1} \psi_{n-1} + V_n \psi_n$$

where $\Phi_{n-1} \in \mathbb{C}^{S_{n-1} \times m}$, $\Upsilon_n \in \mathbb{C}^{S_n \times m}$ are matrices of full rank m for $n \geq 1$.

- Think of Φ_n and Υ_n as a collection of m linear independent (orthogonal) vectors $\Phi_{n,j}, \Upsilon_{n,j} \in \mathbb{C}^{S_n}$, then

$$\Phi_n \Upsilon^*_{n+1} = \sum_{j=1}^{m} |\Phi_{n,j}\rangle \langle \Upsilon_{n+1,j}|$$

Hermitian *m*-channel operators

- As infinite matrix

$$H = \begin{pmatrix} V_0 & -\Phi_0\Upsilon_1^* & & \\ -\Upsilon_1\Phi_0^* & V_1 & -\Phi_1\Upsilon_2^* & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

- In that case, consider $H\psi = z\psi$ which gives

$$\Phi_n\Upsilon_{n+1}^*\psi_{n+1} + \Upsilon_n\Phi_{n-1}^*\psi_{n-1} \;=\; (V_n - z)\psi_n$$

- Let $x_n = \Upsilon_n^*\psi_n$, $\tilde{x}_n = \Phi_n^*\psi_n$, and $z \notin \mathrm{spec}(V_n)$ then it follows

$$(V_n - z)^{-1}\Phi_n x_{n+1} + (V_n - z)^{-1}\Upsilon_n\tilde{x}_n \;=\; \psi_n$$

- Defining $\begin{pmatrix} \alpha_{z,n} & \beta_{z,n} \\ \gamma_{z,n} & \delta_{z,n} \end{pmatrix} = \begin{pmatrix} \Upsilon_n^* \\ \Phi_n^* \end{pmatrix} (V_n - z)^{-1} \begin{pmatrix} \Upsilon_n & \Phi_n \end{pmatrix}$ this gives

$$\beta_{z,n}x_{n+1} + \alpha_{z,n}\tilde{x}_{n-1} \;=\; x_n\;, \quad \delta_{z,n}x_{n+1} + \gamma_{z,n}\tilde{x}_{n-1} \;=\; \tilde{x}_n$$

- $\beta_{z,n} x_{n+1} + \alpha_{z,n} \tilde{x}_{n-1} = x_n \;, \quad \delta_{z,n} x_{n+1} + \gamma_{z,n} \tilde{x}_{n-1} = \tilde{x}_n$
- From there, if $\beta_{z,n}$ is invertible, one may deduce

$$\begin{pmatrix} x_{n+1} \\ \tilde{x}_n \end{pmatrix} = \underbrace{\begin{pmatrix} \beta_{z,n}^{-1} & -\beta_{z,n}^{-1}\alpha_{z,n} \\ \delta_{z,n}\beta^{-1} & \gamma_{z,n} - \delta_{z,n}\beta_{z,n}^{-1}\alpha \end{pmatrix}}_{=: T_n^z} \begin{pmatrix} x_n \\ \tilde{x}_{n-1} \end{pmatrix}$$

- For $n = 0$ we choose any $\Upsilon_0 \in \mathbb{C}^{S_0 \times m}$ of full rank m.

Theorem (S.)

Let $\mu(f) = \Upsilon_0^* f(H) \Upsilon_0$ be the matrix valued spectral measure at Υ_0, let $T_{0,n}^E = T_n^E \cdots T_0^E$, then for $f \in C_b(\mathbb{R})$

$$\int f(E)\,\mathrm{d}\mu(E) = \int f(E)\,\mathrm{d}\nu(E) +$$

$$\lim_{n\to\infty} \int \frac{f(E)}{\pi} \left[\begin{pmatrix} \mathbf{1} & \mathbf{0} \end{pmatrix} \left(T_{0,n}^E \right)^* T_{0,n}^E \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} \right]^{-1} \mathrm{d}E$$

where ν is induced by finitely supported eigenfunctions.

Theorem (S.)

Let $\varphi \in \mathbb{C}^m$ of norm 1, $\phi = \Upsilon_0 \varphi \in \mathbb{C}^{S_0} \subset \ell^2(\mathbb{G})$, $\mu_\phi(f) = \langle \phi | f(H) | \phi \rangle$, then with $\nu_\phi = \varphi^* \nu \varphi$ (ν as before) we have for any $f \in C_b(\mathbb{R})$

$$\int f(E)\, \mu_\phi(\mathrm{d}E) \;=\; \nu_\phi(\mathrm{d}E) \;+\; \lim_{n\to\infty} \int \frac{f(E)\, \mathrm{d}E}{\pi \min\limits_{\varphi^* v = 0} \left\| T^E_{0,n} \begin{pmatrix} \varphi + v \\ 0 \end{pmatrix} \right\|^2}$$

- The connection to the former formula comes from the fact that for $\|x\| = 1$ one has

$$x^*(A^*A)^{-1}x \;=\; \frac{1}{\min\limits_{x^* v = 0} \|A(x+v)\|^2}$$

- Most general form: ranks $r_n = \mathrm{rank}(\Upsilon_n) = \mathrm{rank}(\Phi_{n-1})$ increase, set $r_0 = \mathrm{rank}(\Upsilon_0) = 1$ so Υ_0 is a vector; the transfer matrices become sets $\mathbb{T}^z_n \subset \mathbb{C}^{2r_{n+1} \times 2r_n}$, particularly $\mathbb{T}^z_{0,n} \subset \mathbb{C}^{2r_{n+1} \times 2}$. Then

$$\int f(E)\, \mu_{\Upsilon_0}(\mathrm{d}E) \;=\; \int f(E)\, \nu(\mathrm{d}E) \;+\; \lim_{n\to\infty} \int \frac{f(E)\, \mathrm{d}E}{\pi \min \|\mathbb{T}^E_{0,n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\|^2}$$

Unitary m channel operators

- Think of H as a sum of two Hermitian operators, $H = \mathcal{W} + \mathcal{V}$ where $\mathcal{V}$ is the block-diagonal part $\mathcal{V} = \bigoplus_{n=0}^{\infty} V_n$, and $\mathcal{W}$ the rank m connections, $\mathcal{W} = \sum_{n=1}^{\infty} \sum_{j=1}^{m} \left(|\Phi_{n-1,j}\rangle \langle \Upsilon_{n,j}| + |\Upsilon_{n,j}\rangle \langle \Phi_{n-1,j}| \right)$
- In the unitary case, instead of the sum of two Hermitian operators, we have the product of two unitary operators $\mathcal{U} = \mathcal{W}\mathcal{V}$, $\mathcal{V}$ being a direct sum acting on the shells and $\mathcal{W}$ giving rank m connections between S_n and $S_{n\pm 1}$.

Unitary m-channel operators

- Let $|S_m| \geq 2m$. We call $\mathcal{U} = \mathcal{W}\mathcal{V}$ and $\tilde{\mathcal{U}} = \mathcal{V}\mathcal{W}$ a conjugated pair of unitary m-channel operators if

$$\mathcal{V} = \bigoplus_{n=0}^{\infty} V_n \quad \text{where} \quad V_n \in \mathbb{U}(S_n) \,.$$

$$\mathcal{W}^{(u)} \;=\; u\, e_{(0,-)} e_{(0,-)}^* + P_0 +$$

$$\sum_{n=1}^{\infty} \left(\left(e_{(n-1,+)},\; e_{(n,-)} \right) W_n \begin{pmatrix} e_{(n-1,+)}^* \\ e_{(n,-)}^* \end{pmatrix} + P_n \right)$$

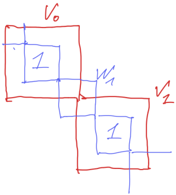
where $e_{(n,\pm)} \in \mathbb{C}^{S_n \times m}$ are such that the column vectors of $Q_n = (e_{(n,-)}, e_{(n,+)}) \in \mathbb{C}^{S_n \times 2m}$ form an orthonormal system in $\mathbb{C}^{S_n}$, $P_n = \mathbf{1}_{S_n} - Q_n Q_n^*$, $u \in \mathbb{U}(m)$ is some sort of 'left boundary condition', and $W_n \in \mathbb{U}(2m)$.

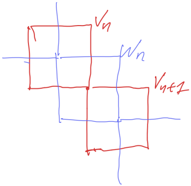
- Using an orthonormal basis of $\mathbb{C}^{S_n}$ where $e_{(n,-)}$ are the first and $e_{(n,+)}$ the last vectors, one may write $\Psi_n = \begin{pmatrix} \Psi_{(n,-)} \\ \Psi_{(n,0)} \\ \Psi_{(n,+)} \end{pmatrix} \in \mathbb{C}^{|S_n|}$ where $\Psi_{(n,\pm)} = e^*_{(n,\pm)} \Psi_n \in \mathbb{C}^m$, $\quad \Psi_{(n,0)} \in \mathbb{C}^{|S_n|-2m}$. In this basis

$$\mathcal{V} = \begin{pmatrix} V_0 & & & \\ & V_1 & & \\ & & V_2 & \\ & & & \ddots \end{pmatrix} \; ; \; \mathcal{W}^{(u)} = \begin{pmatrix} u & & & & & \\ & \mathbf{1}_{|S_0|-2m} & & & & \\ & & W_1 & & & \\ & & & \mathbf{1}_{|S_1|-2m} & & \\ & & & & W_2 & \\ & & & & & \ddots \end{pmatrix}$$

- block overlaps

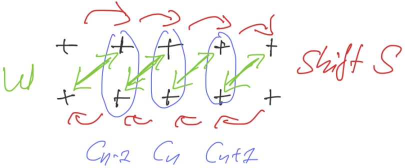
scattering zipper: $\forall n, |S_n| = 2m$





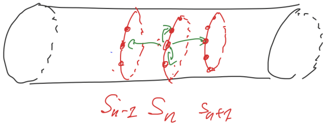
Examples

- Scattering zippers: $\forall n : |S_n| = 2m$
- one-channel scattering zippers include CMV matrices and 1D quantum walks :
  For a 1D quantum walk on a (half) line, consider $\ell^2(\mathbb{Z} \times \{\uparrow, \downarrow\})$, or $\ell^2(\mathbb{Z}_+ \times \{\uparrow, \downarrow\})$, $S_n = \{(n, \uparrow), (n, \downarrow)\}$
  we take a coin operator $\mathcal{C} = \bigoplus_n C_n$, $C_n \in \mathbb{U}(S_n)$, and the shift $\mathcal{S}\delta_{n,\uparrow} = \delta_{n+1,\uparrow}$, $\mathcal{S}\delta_{n,\downarrow} = \delta_{n-1,\downarrow}$. Then the walk is given by $\mathcal{U} = \mathcal{SC}$.
- Define $\mathcal{W}$ to interchange $(n, \downarrow)$ with $(n+1, \uparrow)$ and $\mathcal{V} = \mathcal{WSC}$, then $\mathcal{U} = \mathcal{SC} = \mathcal{WWSC} = \mathcal{WV}$ is in the form of a one-channel operator.



Examples

- 2D type quantum walks on cylinders $\mathbb{Z} \times \mathbb{Z}_m$ or $\mathbb{Z}_+ \times \mathbb{Z}_m$ (using 4 spins going in 4 directions)
  Here one finds $|S_n| = 4m$ and the connections are of rank m, meaning these models are not included within scattering zippers, but they are unitary m channel operators.



- Quantum walks on structures like nano tubes or (infinite) carbon-chains

Proposition (Marin, Schulz-Baldes)

With $\mathcal{U}^{(u)} = \mathcal{W}^{(u)}\mathcal{V}$, $\tilde{\mathcal{U}}^{(u)} = \mathcal{V}\mathcal{W}^{(u)}$, the following sets of equations are equivalent (in fact for solutions $\Psi, \Phi \in \mathbb{C}^{\mathbb{G}}$)

1. $\mathcal{U}^{(u)}\Psi = z\Psi \ \wedge \ \mathcal{W}^{(u)}\Phi \ = \ \Psi$
2. $\mathcal{V}\Psi = z\Phi \ \wedge \ \mathcal{W}^{(u)}\Phi \ = \ \Psi$
3. $\tilde{\mathcal{U}}^{(u)}\Phi = z\Phi \wedge \ \mathcal{W}^{(u)}\Phi \ = \ \Psi.$

- From (ii) one can deduce

$$\Psi_{(0,-)} = u\Phi_{(0,-)}\ , \quad \Psi_{(n,0)} = \Phi_{(n,0)}\ , \quad \begin{pmatrix} \Psi_{(n-1,+)} \\ \Psi_{(n,-)} \end{pmatrix} \ = \ W_n \begin{pmatrix} \Phi_{(n-1,+)} \\ \Phi_{(n,-)} \end{pmatrix}$$

$$\begin{pmatrix} \Psi_{(n,-)} \\ \Psi_{(n,+)} \end{pmatrix} \ = \ Q_n^*(z^{-1}V_n - P_n)^{-1}Q_n \begin{pmatrix} \Phi_{(n,-)} \\ \Phi_{(n,+)} \end{pmatrix}$$

For $|z| = 1$ one finds that $Q_n^*(z^{-1}V_n - P_n)^{-1}Q_n$ is unitary.

Proposition (Marin, Schulz-Baldes; S.)

For a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{C}^{2m\times 2m}$ where β is invertible define

$$\varphi_\sharp(M) = \begin{pmatrix} \beta^{-1} & -\beta^{-1}\alpha \\ \delta\beta^{-1} & \gamma - \delta\beta^{-1}\alpha \end{pmatrix} \quad \text{and} \quad \varphi_\flat(M) = \begin{pmatrix} \gamma - \delta\beta^{-1}\alpha & \delta\beta^{-1} \\ -\beta^{-1}\alpha & \beta^{-1} \end{pmatrix} .$$

Then,

$$\begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix} = M \begin{pmatrix} \Phi_- \\ \Phi_+ \end{pmatrix} \Leftrightarrow \begin{pmatrix} \Phi_+ \\ \Psi_+ \end{pmatrix} = \varphi_\sharp(M) \begin{pmatrix} \Psi_- \\ \Phi_- \end{pmatrix} \Leftrightarrow \begin{pmatrix} \Psi_+ \\ \Phi_+ \end{pmatrix} = \varphi_\flat(M) \begin{pmatrix} \Phi_- \\ \Psi_- \end{pmatrix} .$$

Moreover, if $M \in \mathbb{U}(2m)$, then $\varphi_\sharp(M), \varphi_\flat(M) \in \mathbb{U}(m,m)$, where

$$\mathbb{U}(m,m) = \left\{ T \in \mathbb{C}^{2\times 2} \,:\, T^* \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\}$$

A side remark: If $M \in \mathrm{Her}(2m)$, then $\varphi_\sharp(M), \varphi_\flat(M)$ are Hermitian symplectic.

Transfer matrices

- From there we get

$$\begin{pmatrix} \Phi_{(n,+)} \\ \Psi_{(n,+)} \end{pmatrix} = T^{\sharp}_{z,n} \begin{pmatrix} \Psi_{(n,-)} \\ \Phi_{(n,-)} \end{pmatrix}, \quad T^{\sharp}_{z,n} = \varphi_{\sharp}\big(Q_n^*(z^{-1}V_n - P_n)^{-1}Q_n\big)$$

$$\begin{pmatrix} \Psi_{(n,-)} \\ \Phi_{(n,-)} \end{pmatrix} = T^{\flat}_{n} \begin{pmatrix} \Phi_{(n-1,+)} \\ \Psi_{(n-1,+)} \end{pmatrix}, \quad T^{\flat}_{n} = \varphi_{\flat}(W_n) = \varphi_{\sharp}(W_n^*)$$

and we define the transfer matrices

$$T_{z,n} = T^{\sharp}_{z,n} T^{\flat}_{n} \quad \text{to get} \quad \begin{pmatrix} \Phi_{(n,+)} \\ \Psi_{(n,+)} \end{pmatrix} = T_{z,n} \begin{pmatrix} \Phi_{(n-1,+)} \\ \Psi_{(n-1,+)} \end{pmatrix} .$$

For $n = 0$ we let $T^{\flat}_0 = \mathbf{1}$

- Note for $|z| = 1$ we have $T_{z,n} \in \mathbb{U}(m, m)$.

Theorem

Let $m = 1$ and let μ be the spectral measure of $\mathcal{U}^{(u)}$ at $e_{(0,-)}$, then, there is a pure point measure ν induced by eigenvectors of compact support, such that for $f \in C(S^1)$,

$$\int_{S^1} f(z)\mathrm{d}\mu(z) = \int_{S^1} f(z)\mathrm{d}\nu(z) +$$

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{f(e^{i\varphi})\,\mathrm{d}\varphi}{\pi \left\| T_{e^{i\varphi},n} T_{e^{i\varphi},n-1} \cdots T_{e^{i\varphi},0} \left(\begin{smallmatrix} u \\ 1 \end{smallmatrix}\right) \right\|^2}$$

In case $m > 1$ we have some positive $m \times m$ matrix valued measure and letting $T_{z,0,n} = T_{z,n} T_{z,n-2} \cdots T_{z,0}$ we get

$$\int_{S^1} f(z)\mathrm{d}\mu(z) = \int_{S^1} f(z)\mathrm{d}\nu(z) +$$

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{f(e^{i\varphi})}{\pi} \left(\left(u^* \ \mathbf{1} \right) T^*_{e^{i\varphi},0,n} T_{e^{i\varphi},0,n} \left(\begin{smallmatrix} u \\ \mathbf{1} \end{smallmatrix}\right) \right)^{-1} \mathrm{d}\varphi$$

Key relation to Green's function

$$Q_{0,N} = \begin{pmatrix} e_{(0,-)} & e_{(N,+)} \end{pmatrix} \in \mathbb{C}^{\mathbb{G}_N \times 2}, \quad P_{0,N} = \mathbf{1}_{\mathbb{G}_N} - Q_{0,N} Q_{0,N}^* \in \mathbb{C}^{\mathbb{G}_N \times \mathbb{G}_N}$$

$$R_{z,[0,N]}^{(u,v)} = Q_{0,N}^* (z^{-1} \mathcal{U}_N^{(u,v)} - \mathbf{1}_{\mathbb{G}_N})^{-1} Q_{0,N}$$

Proposition

For any N and any $u, v, z \in \mathbb{C}$ where all quantities are well defined, we have

$$\varphi_\sharp(R_{z,[0,N]}^{(u,v)}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} T_{z,[0,N]} \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\varphi_\sharp(\tilde{R}_{z,[0,N]}^{(u,v)}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v^{-1} \end{pmatrix} T_{z,[0,N]} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Criteria for a.c. spectrum

Corollary

Assume we find $\zeta_n(\varphi) \in \mathbb{C}^m$ *and* $p > 1$ *such that*

$$\liminf_{n\to\infty} \int_a^b \left\| T_{e^{i\varphi},0,n} \begin{pmatrix} u\zeta_n(\varphi) \\ \zeta_n(\varphi)+\chi \end{pmatrix} \right\|^{2p} \mathrm{d}\varphi < \infty$$

Then, the measure $\chi^*(\mu - \nu)\chi$ (*part of spectral measure at* $e_{(0,-)}\chi$) *is purely absolutely continuous in* $e^{i(a,b)}$*, with respect to the Haar measure on* S^1.

Corollary

Let be given a periodic scattering zipper $\mathcal{U} = \mathcal{W}\mathcal{V}$, *with* $V_n = \begin{pmatrix} \mathfrak{a}_n & \mathfrak{b}_n \\ \mathfrak{c}_n & \mathfrak{d}_n \end{pmatrix}, W_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \in \mathbb{U}(2),\ V_{n+p} = V_n,\ W_{n+p} = W_n,$ $b_n \neq 0,\ \mathfrak{b}_n \neq 0$. *Let* $T_z = T_{z,1,p}$ *be the transfer matrix for the operator* $\mathcal{U}$ *over one period and let* $\Sigma = \{z \in S^1 \,:\, |\mathrm{Tr}(T_z)| < 2\}$.

Let $\widehat{\mathcal{U}}_\omega = \widehat{\mathcal{W}}_\omega \widehat{\mathcal{V}}_\omega$ *be a random perturbation where*

- *The family of pairs* $(\widehat{V}_n, \widehat{W}_n)_n$ *is independent*
- $\sum_n \left[\|\mathbb{E}(\widehat{V}_n - V_n)\| + \|\mathbb{E}(\widehat{W}_n - W_n)\| + \mathbb{E}(\|\widehat{V}_n - V_n\|^2) + \mathbb{E}(\|\widehat{W}_n - W_n\|^2) \right] < \infty$
- $\exists\, \varepsilon > 0\ \forall\, n \in \mathbb{Z}_+ \,:\, |b_n| > \varepsilon \wedge |\mathfrak{b}_n| > \varepsilon$ *almost surely.*

Then

a. $\sigma_{ess}(\mathcal{U}) = \overline{\Sigma}$, *and almost surely,* $\sigma_{ess}(\widehat{\mathcal{U}}_\omega) = \overline{\Sigma}$.
b. *The spectrum of* $\mathcal{U}$ *and, almost surely, the spectrum of* $\widehat{\mathcal{U}}_\omega$ *are purely absolutely continuous in* Σ.

Proof ideas

- Connect products of first n transfer matrices to Green's function and Poisson kernel of spectral measure for finite pieces of the operator $\mathcal{U}$. Due to the set-up and "grouping" of shells to bigger shells, formulas only need to be proved for $n = 1$.
- Take a spectral average over a right boundary condition v and realize that by strong resolvent convergence, for $n \to \infty$ the averaged measures converge weakly to the actual spectral measure
- On the level of the Poisson transform this corresponds to replacing v by $\mathbf{0}$.
- Then, we compute the density of the absolutely continuous part and analyze the singular part.

some future directions

- Transfer matrices and spectral averaging formula for general finite hopping unitary operators
- Deift-Killip type Theorems for quantum walks on antitrees and operators with radial symmetry; sum rules
- Absolutely continuous spectrum for random l^2 perturbed coins on the strip, cylindrical structures, nano tubes

References

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THANK YOU!!