Bures geodesics on the manifold of quantum states

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Outlines

- Geodesics on the manifold of quantum states
- Geodesics as non-Markovian physical evolutions
- Ideas of the proofs of Theorems 1 and 2
- Uhlmann holonomy
- Conclusions & perspectives

Joint work with Sergio Carrasco, UdeC

Manifold of mixed quantum states $\mathcal{E}_{\mathcal{H}}$

- A **mixed state** of a quantum system is given by a non-negative trace-class operator $\rho : \mathcal{H} \to \mathcal{H}$ with trace one (density matrix) $\mathcal{H}$ = system Hilbert space. In what follows: $\dim \mathcal{H} = n < \infty$

- Open manifold of invertible mixed states

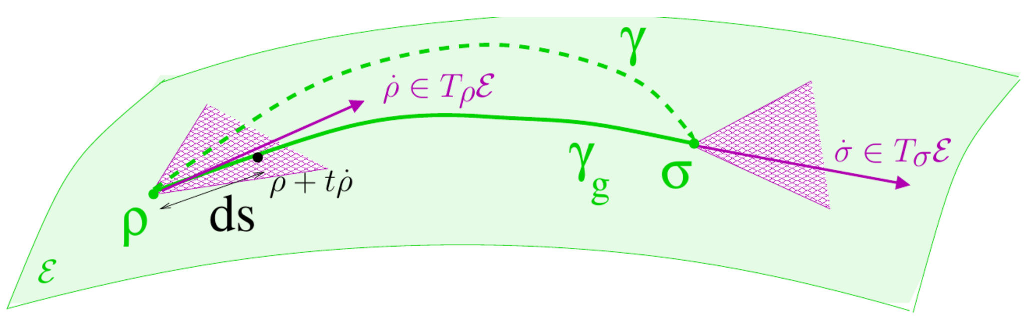
$$\mathcal{E}_{\mathcal{H}}^{\mathrm{inv}} = \{\rho : \mathcal{H} \to \mathcal{H}\, ;\, \rho > 0, \operatorname{tr} \rho = 1\}$$

  Boundary $\partial\mathcal{E}_{\mathcal{H}} = \{$non-invertible $\rho \geqslant 0, \operatorname{tr}\rho = 1\}$
  $e.g.$ pure state $\rho_\psi = |\psi\rangle\langle\psi| \in \partial\mathcal{E}_{\mathcal{H}}$.

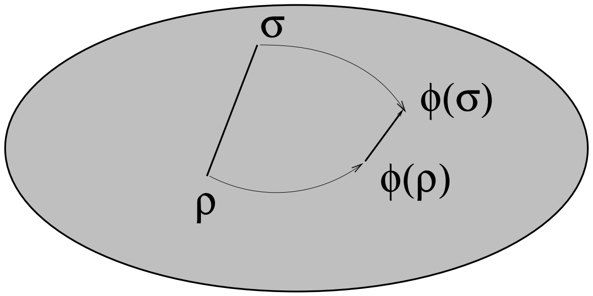
- Tangent space $T_\rho\mathcal{E}_{\mathcal{H}}$ at ρ

$$T_\rho\mathcal{E}_{\mathcal{H}} = \{\dot\rho = \dot\rho^*\, ;\, \operatorname{tr}\dot\rho = 0\}$$

  (real) vector space of self-adjoint traceless op. on $\mathcal{H}$.



Contractive distances on $\mathcal{E}_{\mathcal{H}}$



CONTRACTIVE DISTANCE

- The manifold of quantum states $\mathcal{E}_{\mathcal{H}}$ can be equipped with many distances $d : \mathcal{E}_{\mathcal{H}} \times \mathcal{E}_{\mathcal{H}} \to \mathbb{R}_+$.

- From a QI point of view, interesting distances must be contractive under CPTP maps, i.e. for any such map Φ on $\mathcal{E}_{\mathcal{H}}$, $\forall\ \rho, \sigma \in \mathcal{E}_{\mathcal{H}}$,

$$d(\Phi(\rho), \Phi(\sigma)) \leqslant d(\rho, \sigma)$$

  Physically: *irreversible evolutions can only decrease the distance between two states.*

- A contractive distance is in particular unitarily invariant, i.e. $d(U\rho U^*, U\sigma U^*) = d(\rho, \sigma)$ for any unitary U on $\mathcal{H}$

- The L^p-distances $d_p(\rho, \sigma) = \|\rho - \sigma\|_p = (\mathrm{tr}\,|\rho - \sigma|^p)^{1/p}$ are *not* contractive excepted for $p = 1$ (trace distance) *[Ruskai '94]*.

Petz's characterization of contractive distances

- **Classical setting:** there exists a unique (up to a multiplicative factor) contractive Riemannian distance $d_{\rm clas}$ on the probability simplex $\mathcal{E}_{\rm clas}$, with Fisher metric $ds^2 = \sum_k dp_k^2/p_k$ *[Cencov '82]*

- **Quantum generalization:** any Riemannian contractive distance on the set of states $\mathcal{E}_{\mathcal{H}}$ with $n = \dim \mathcal{H} < \infty$ has metric

$$g_\rho(\dot\rho, \dot\rho) = \sum_{k,l=1}^{n} c(p_k, p_l) |\langle k|\dot\rho|l\rangle|^2$$

where p_k and $|k\rangle$ are the eigenvalues and eigenvectors of ρ,

$$c(p,q) = \frac{p f(q/p) + q f(p/q)}{2pq f(p/q) f(q/p)}$$

and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an arbitary operator-monotone function such that $f(x) = x f(1/x)$ *[Morozova & Chentsov '90, Petz '96]*

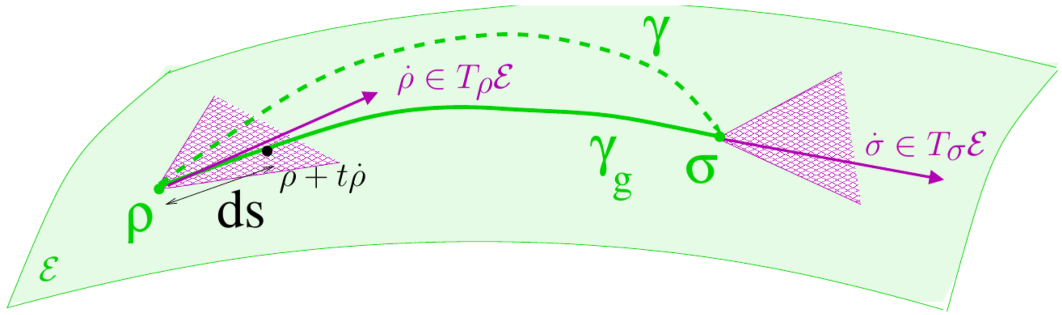
Bures distance

- **Fidelity** (generalizes $F = |\langle\psi|\phi\rangle|^2$ for mixed states) *[Uhlmann '76]*

$$F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = \left(\mathrm{tr}[\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}]\right)^2 = F(\sigma,\rho)$$

- **Bures distance:** $d_{\mathrm{Bu}}(\rho,\sigma) = \left(2 - 2\sqrt{F(\rho,\sigma)}\right)^{\frac{1}{2}}$ *[Bures '69]*
  - $\hookrightarrow$ has metric of the Petz form with $f(x) = \frac{x+1}{2}$
  - $\hookrightarrow$ **smallest** contractive Riemannian distance *[Petz '96]*
  - $\hookrightarrow$ coincides with the Fubiny-Study metric on $P\mathcal{H}$ for pure states
  - $\hookrightarrow$ $d_{\mathrm{Bu}}(\rho,\sigma)^2$ is jointly convex in (ρ,σ).

Bures metric and arccos distance



- **Bures metric:** $\mathrm{d}s^2_{\mathrm{Bu}} = d_{\mathrm{Bu}}(\rho, \rho + t\dot\rho)^2 = (g_{\mathrm{Bu}})_\rho(\dot\rho, \dot\rho)t^2 + o(t^2)$

- Geodesic distance for the Bures metric: **arccos distance**

$$\boxed{\theta_{\mathrm{Bu}}(\rho, \sigma) = \min_{\gamma:\rho\to\sigma} \ell(\gamma) = \arccos\sqrt{F(\rho, \sigma)}}$$

(min over all smooth curves $\gamma : [0,1] \to \mathcal{E}_{\mathcal{H}}$, $\gamma(0) = \rho$, $\gamma(1) = \sigma$)

$\hookrightarrow$ θ_{Bu} has same metric as $d_{\mathrm{Bu}}(\rho, \sigma) = 2\sin(\theta_{\mathrm{Bu}}(\rho, \sigma)/2)$.

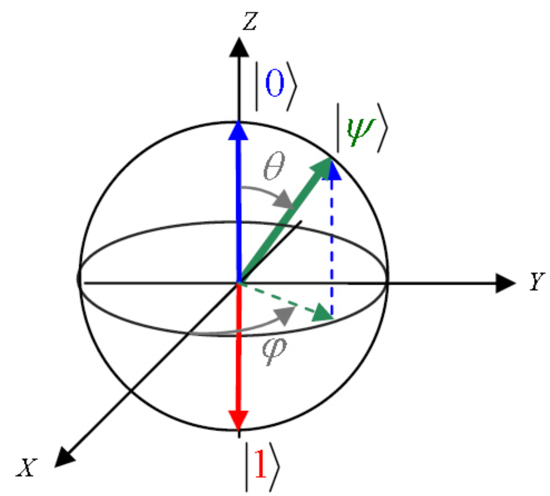
- **Geodesics:** smooth curves $\gamma_{\mathrm{g}} : [0, \theta] \mapsto \mathcal{E}_{\mathcal{H}}$ with constant velocity minimizing the length *locally*.

Example: one qubit

For a qubit ($\mathcal{H} = \mathbb{C}^2$), quantum states can be parametrized by the Bloch vector $\mathbf{r}$ as

$$\rho = \rho(\mathbf{r}) = \frac{1}{2}\big(\mathbb{1} + \mathbf{r}\cdot\vec{\sigma}\big)\ ,\ |\mathbf{r}| \leqslant 1\,.$$

with $\sigma_1, \sigma_2, \sigma_3 =$ Pauli matrices.



Bures metric

$$ds_{\mathrm{B}}^2 = (g_{\mathrm{B}})_{\alpha\beta}\mathrm{d}\alpha\mathrm{d}\beta = \frac{1}{4}\Big(\mathrm{d}t^2 + \sin^2 t\big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\big)\Big)$$

$t = \arcsin(|\vec{r}|) \in [0, \pi/2]$, $\theta, \varphi =$ spherical coordinate angles

$\hookrightarrow$ **metric of the (half) 3-sphere $S^3 \subset \mathbb{R}^4$.**

Consequences: (i) $\mathcal{E}_{\mathcal{H}}^{\mathrm{inv}}$ has a constant positive curvature.

(ii) the geodesics are projections of great circles of S^3 on the $\mathbf{r} = (x, y, z)$-hyperplane

Determination of Bures geodesics

Consider a system with Hilbert space dimension $n < \infty$.
Let ρ and $\sigma \in \mathcal{E}_{\mathcal{H}}$ be two invertible mixed states.

TH 1: To any unitary self-adjoint operator V commuting with $\Lambda_{\sigma\rho} = (\sqrt{\rho}\,\sigma\sqrt{\rho})^{1/2}$, there corresponds a geodesic arc $\rho \to \sigma$:

$$\gamma_{\mathrm{g},V}(\tau) = X_{\rho\sigma,V}(\tau)\,\rho\,X_{\rho\sigma,V}(\tau) \quad , \quad 0 \leqslant \tau \leqslant \theta_V \ ,$$

$$X_{\rho\sigma,V}(\tau) = \frac{1}{\sin\theta_V}\Big(\sin(\tau)\rho^{-1/2}\Lambda_{\sigma\rho}V\rho^{-1/2} + \sin(\theta_V - \tau)\mathbb{1}\Big)$$

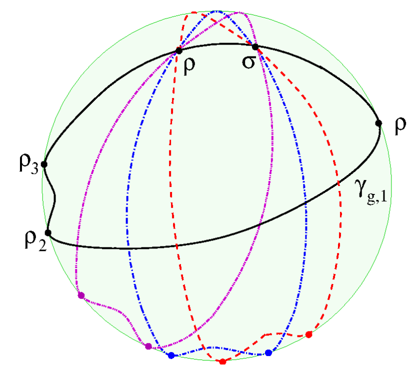
with θ_V = geodesic length = $\arccos(\mathrm{tr}[\Lambda_{\sigma\rho}V])$

$\hookrightarrow$ there are $N = 2^n$ (resp. $N = \infty$) geodesic arcs joining ρ and σ if $\Lambda_{\sigma\rho}$ has nondegenerated (resp. degenerate) spectrum.

$\hookrightarrow$ the shortest geodesic arc is obtained for $V = \mathbb{1}$.
It has length $\theta_1 = d_{\mathrm{B}}(\rho,\sigma)$

Previous works: $V = \mathbb{1}$ [Ericsson, J.Phys.A '05; Barnum, PhD thesis '98]

Intersections with the boundary



$n = 3$

- Boundary of quantum states $\partial\mathcal{E}_{\mathcal{H}} = \{\text{non-invertible } \rho \geqslant 0, \operatorname{tr}\rho = 1\}$

- The number of intersections of $\gamma_{\mathrm{g},V}$ with the boundary $\partial\mathcal{E}_{\mathcal{H}}$ between ρ and σ is equal to the multiplicity of the eigenvalue -1 of V. (note that $\operatorname{spec}(V) = \{1, -1\}$ since V is unitary & s.a.). In particular, the shortest geodesic $\gamma_{\mathrm{g},\mathbb{1}}$ does not interest $\partial\mathcal{E}_{\mathcal{H}}$ between ρ and σ.

- By prolongating $\gamma_{\mathrm{g},V}$: closed geodesics intersecting q_V times $\partial\mathcal{E}_{\mathcal{H}}$, with $q_V = \sharp$ of distinct eigenvalues of $M_{\rho\sigma,V}$. The intersection states ρ_i have ranks $n - m_{i,V}$ and supports $(\mathbb{1} - P_{i,V})\mathcal{H}$, with $m_{i,V}, P_{i,V}$ = multiplicities and spectral projectors of $M_{\rho\sigma,V}$ *[Ericsson, J.Phys.A '05]*

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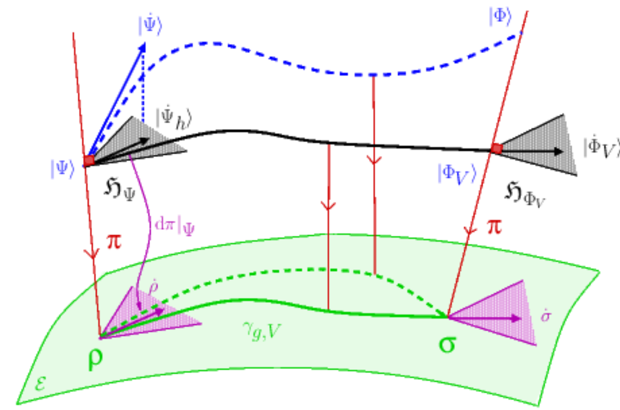
Purifications and horizontal lifts

▻ Consider an ancilla (environment) with Hilbert space $\mathcal{H}_A$, $\dim \mathcal{H}_A \geqslant n$. A purification of the system state $\rho \in \mathcal{E}_{\mathcal{H}}$ is a pure state $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}_A$ such that

$$\rho = \mathrm{tr}_A |\Psi\rangle\langle\Psi| = \pi(|\Psi\rangle)$$

▻ Purifications of ρ are not unique: given a purification $|\Psi_0\rangle$, all other purifications are in the *orbit of ρ under the unitary group action on $\mathcal{H}_A$:*



$$\pi^{-1}(\rho) = \{|\Psi\rangle = \mathbb{1} \otimes U_A |\Psi_0\rangle \, ; \, U_A \text{ unitary on } \mathcal{H}_A\}$$

▻ Horizontal lift of a curve $\gamma : [0,\theta] \to \mathcal{E}_{\mathcal{H}}$: $t \in [0,\theta] \mapsto |\Psi_{\mathrm{h}}(t)\rangle$,

$$\gamma(t) = \pi(|\Psi_{\mathrm{h}}(t)\rangle) \ , \ |\dot{\Psi}_{\mathrm{h}}(t)\rangle \in \mathfrak{H}_{|\Psi_{\mathrm{h}}(t)\rangle} \text{ horizontal subspace}$$

Geodesic Hamiltonian

- **TH 2:** There is a Hamiltonian $H_{\mathrm{g},V}$ on $\mathcal{H} \otimes \mathcal{H}_A$ s.t.

$$\boxed{\gamma_{\mathrm{g},V}(\tau) = \mathrm{tr}_A\left(e^{-\mathrm{i}\tau H_{\mathrm{g},V}}|\Psi\rangle\langle\Psi|e^{\mathrm{i}\tau H_{\mathrm{g},V}}\right)}$$

  where $|\Psi\rangle$ is a purification of ρ. This Hamiltonian is

$$H_{\mathrm{g},V} = -\mathrm{i}\big(|\Psi\rangle\langle\dot{\Psi}_V| - |\dot{\Psi}_V\rangle\langle\Psi|\big)$$

  where $|\dot{\Psi}_V\rangle$ is a normalized vector orthogonal to $|\Psi\rangle$ satisfying the *horizontality condition:*

$$|\dot{\Psi}_V\rangle \in \mathfrak{H}_{|\Psi\rangle} = \{L \otimes \mathbb{1}_A|\Psi\rangle\,;\, L \text{ self-adjoint }, \langle L \otimes \mathbb{1}_A\rangle_\Psi = 0\}$$

  $\hookrightarrow$ Geodesics correspond to physical evolutions of the system coupled to an ancilla.

- In particular, if $\gamma_{\mathrm{g},V}$ passes through a pure state $\rho = \rho_\psi$, the initial system-ancilla state is decorrelated, $|\Psi\rangle = |\psi\rangle \otimes |\alpha\rangle$.
  NOTE: always the case for a qubit case ($n = 2$).

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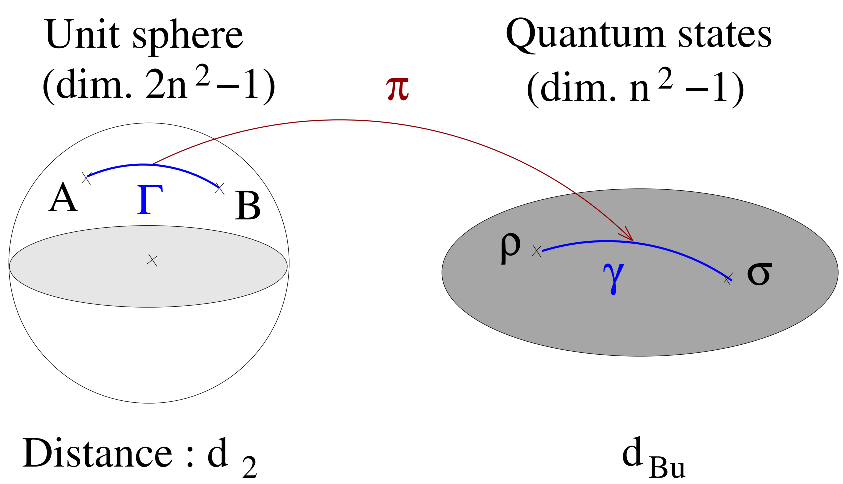
Uhlmann fiber bundle



Consider the unit sphere $\mathcal{S}$ of the normalized vectors $|\Psi\rangle \in \mathcal{H}\otimes\mathcal{H}_A$, with $\dim \mathcal{H}_A = \dim \mathcal{H} = n$, equipped with the norm distance

$$d_{\mathcal{S}}(|\Psi\rangle, |\Phi\rangle) = \|\Psi - \Phi\|$$

★ **Projection map:** $\pi : \mathcal{S} \to \mathcal{E}_{\mathcal{H}}$

$$\pi(|\Psi\rangle) = \operatorname{tr}_A |\Psi\rangle\langle\Psi|$$

Orbit of ρ under the unitary group action:

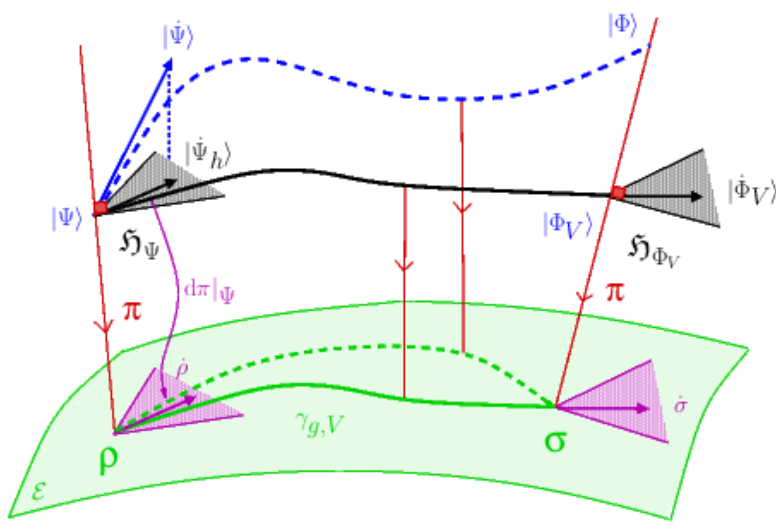
$$\pi^{-1}(\rho) = \{|\Psi\rangle = \mathbb{1} \otimes U_A|\Psi_0\rangle\,;\, U_A \text{ unitary on } \mathcal{H}_A\}$$

★ Then $\mathcal{E}_{\mathcal{H}} = \mathcal{S}/U(n)$. The Bures distance on $\mathcal{E}_{\mathcal{H}}$ is *[Uhlmann '76]*

$$d_{\mathrm{Bu}}(\rho, \sigma) = \inf_{|\Phi\rangle \in \pi^{-1}(\sigma)} d_{\mathcal{S}}(|\Psi_0\rangle, |\Phi\rangle)$$

Riemannian submersions and geodesics

★ The map π from $(\mathcal{S}^{\,\mathrm{inv}}, d_{\mathcal{S}})$ to $(\mathcal{E}_{\mathcal{H}}^{\mathrm{inv}}, d_{\mathrm{B}})$ is a smooth Riemannian submersion, i.e. its differential $D\pi|_{|\Psi\rangle}$ is an isometry $[\ker(D\pi|_{|\Psi\rangle})]^{\perp} \to T_{\rho}\mathcal{E}_{\mathcal{H}}$ for all $|\Psi\rangle \in \mathcal{S}$.



★ **TH:** A smooth Riemannian submersion $\pi : \mathcal{S}^{\,\mathrm{inv}} \to \mathcal{E}_{\mathcal{H}}^{\mathrm{inv}}$ maps geodesics of $(\mathcal{S}^{\,\mathrm{inv}}, d_{\mathcal{S}})$ with horizontal initial tangent vectors $|\dot{\Psi}_{\mathrm{h}}\rangle \in [\ker(D\pi|_{|\Psi\rangle})]^{\perp}$ onto geodesics in $(\mathcal{E}_{\mathcal{H}}^{\mathrm{inv}}, d_{\mathrm{B}})$.

★ $\mathcal{S}^{\,\mathrm{inv}}$ is induced by the Euclidean metric $d_{\mathcal{S}}$ on $\mathcal{H} \otimes \mathcal{H}_A$
$\hookrightarrow$ the geodesics $|\Psi_{\mathrm{g},V}(\tau)\rangle$ on $\mathcal{S}^{\,\mathrm{inv}}$ are great circles.

★ Using these fact, one finds the explicit form of the Bures geodesics and obtains that $|\Psi_{\mathrm{g},V}(\tau)\rangle = e^{-\mathrm{i}\tau H_{\mathrm{g},V}}|\Psi\rangle$.

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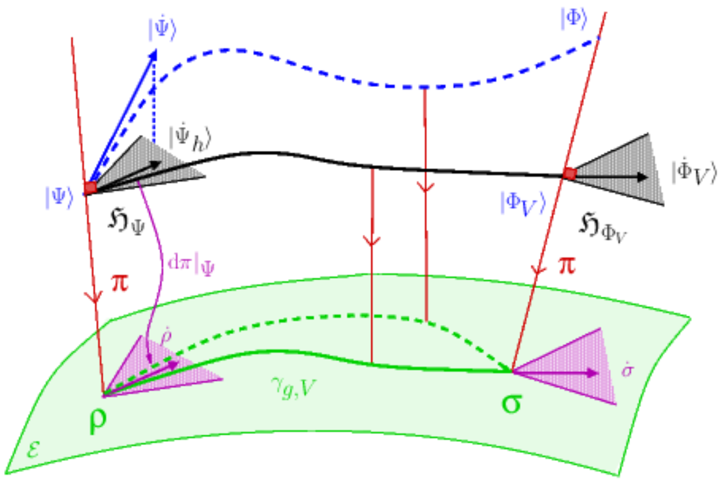
Uhlmann holonomy



- For a given purification $|\Psi\rangle \in \mathcal{H}\otimes\mathcal{H}_A$ of ρ, let

$$\tau \in [0,\pi] \mapsto |\Psi^{\rm g}_{\rm h}(\tau)\rangle$$

be a horizontal geodesic on $\mathcal{H}\otimes\mathcal{H}_A$ starting at $|\Psi\rangle$. (Uhlmann parallel transport).

- Given two ONB $\{|k\rangle\}_{k=1}^n$ of $\mathcal{H}$ and $\{|\alpha_l\rangle\}_{l=1}^n$ of $\mathcal{H}_A$, let

$$\tau \mapsto |\Psi^{\rm g}_{\rm ref}(\tau)\rangle = \sqrt{\gamma_{{\rm g},V}(\tau)} \otimes 1_A \sum_{k=1}^n |k\rangle \otimes |\alpha_k\rangle$$

be a *reference lift* of the geodesic $\gamma_{{\rm g},V}$ s.t. $|\Psi^{\rm g}_{\rm ref}(0)\rangle = |\Psi\rangle$.

- Uhlmann holonomy: unitary operator $U_A(t)$ on $\mathcal{H}_A$ such that

$$|\Psi^{\rm g}_{\rm h}(\tau)\rangle = 1 \otimes U_A(\tau)|\Psi^{\rm g}_{\rm ref}(\tau)\rangle$$

Uhlmann phase of geodesics

- Let $U_{\sigma\rho}$ be the unitary operator in the polar decomposition
$$\sqrt{\sigma}\sqrt{\rho} = U_{\sigma\rho}\Lambda_{\sigma\rho} \quad , \quad \Lambda_{\sigma\rho} = |\sqrt{\sigma}\sqrt{\rho}| \; .$$
Then $\quad U_A(\tau) = (U_{\gamma_{\mathrm{g}}(\tau)\rho} V)^T$

  $V=$ unitary defining the geodesic, $O^T = \sum_{k,l} \langle l|O|k\rangle \, |\alpha_k\rangle\langle\alpha_l|$

- Uhlmann geometric phase: $\phi_{\mathrm{U}}(\tau) = \arg\langle\Psi|\Psi_{\mathrm{h}}^{\mathrm{g}}(\tau)\rangle$ (independent of the purification $|\Psi\rangle$ of ρ). One finds
$$\phi_{\mathrm{U}}(\tau) = \mathrm{tr}(\Lambda_{\gamma_{\mathrm{g}}(\tau)\rho} V) = \begin{cases} 0 & \text{if } 0 \leqslant \tau < \pi/2 \\ \pi & \text{if } \pi/2 < \tau \leqslant \pi. \end{cases}$$
$\hookrightarrow$ *topological kick at* $\tau = \pi/2$, *i.e. when* $|\Psi_{\mathrm{h}}^{\mathrm{g}}(\tau)\rangle \perp |\Psi\rangle$.

- In particular, the Uhlmann phase of any closed geodesic is non-trivial, $\phi_{\mathrm{U}}(\pi) = \pi$. Indeed, $|\Psi_{\mathrm{h}}^{\mathrm{g}}(k\pi)\rangle = (-1)^k|\Psi\rangle$, $k = 1, 2$.

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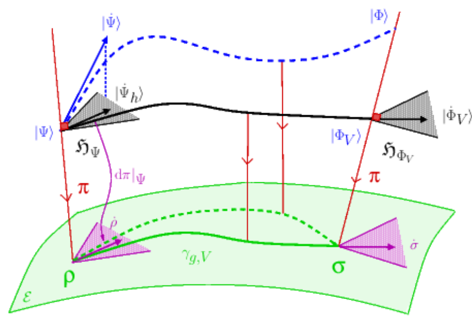
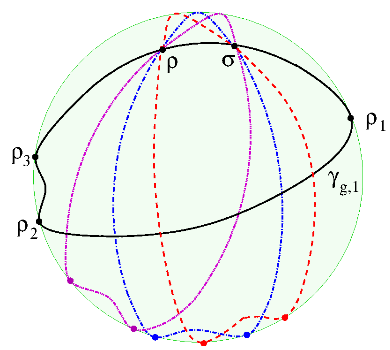
Conclusions

- Explicit form of the Bures geodesics on the manifold of quantum states
  - ↪ *generalizes previous works by A. Ericsson and H.N. Barnum.*
  - ↪ *relies on the theory of Riemannian submersions.*



- The geodesics can be realized as physical evolutions of the system coupled to an ancilla
  - ↪ *can be simulated in experiments and quantum computers!*
- The Uhlmann phase of closed geodesics is non trivial and exhibits a topological kick.
  - ↔ *the shortest geodesic arc between 2 invertible states, which does not intersect $\partial\mathcal{E}_{\mathcal{H}}$, has a trivial phase.*

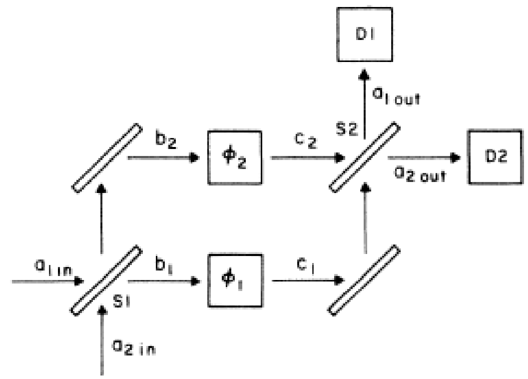


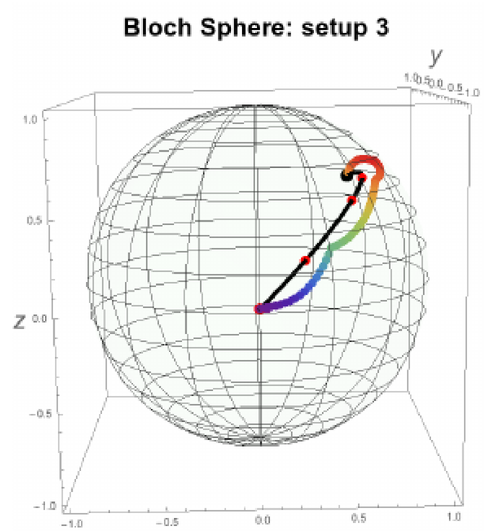
$n = 3$

THANK YOU FOR YOUR ATTENTION!

Why are Bures geodesics interesting?

★ Applications to **quantum metrology**:
Estimate some unknown parameter(s) ϕ *from measurements on the output state of a* ϕ*-dependent quantum channel.*
$\hookrightarrow$ Bures metric $g_{\rm B}(\dot{\rho},\dot{\rho}) =$ *Quantum Fisher Information* giving the best possible precision in the estimation.





★ Applications to **quantum control**:
Steering an initial state to a target state using a given time-dependent Hamiltonian / Liouvillian depending on some control parameters.

★ Relation with the **Quantum speed limit**.