Topological classification of states of the Weyl C*-algebra

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FisMat Days
December 2024

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Observables in Physics

There are several relevant physical quantities associated to a physical system that can be measured in experiments.

Examples: Energy, momentum, position, kinetic energy, potential energy, force, Hamiltonian, Lagrangian, temperature, pressure...

In classical mechanics, these *observable* quantities are described by continuous functions $(x, p) \mapsto f(x, p)$.

In quantum mechanics, they are described by (bounded) self-adjoint operators in a Hilbert space $A : \mathcal{H} \to \mathcal{H}$.

Both of these objects are examples of elements of *C^*-algebras*.

Some context

- The quantum nature of a physical system is encoded in the Canonical Commutation Relations (CCR) of position (q) and momentum (p):

$$[q, p] = \mathrm{i}\hbar\mathbf{1}.$$

- More abstractly: The C^*-algebraic formulation of QM encodes the CCR's in the *Weyl C^*-algebra* $\mathscr{W}$.
- Usual setting of QM in $L^2(\mathbb{R}^d)$: Schrödinger representation of $\mathscr{W}$.
- Stone-Von Neumann: any regular irreducible representation of $\mathscr{W}$ is unitarily equivalent to the Schrödinger representation.
- Most times this is seen as enough.

Some context

- In this sense, $\mathscr{W}$ usually amounts to a stepping stone to the "analytical" Schrödinger quantum mechanics.
- It is not "big enough" of a setting: no spectral theorems and many Hamiltonian evolutions do not preserve it [FV][1].
- Hence, one usually works in the enveloping von Neumann algebra $\rho(\mathscr{W})''$ associated to a representation ρ of $\mathscr{W}$.
- Since $\mathscr{W}$ is simple, any state of $\rho(\mathscr{W})''$ restricts to a state of $\mathscr{W}$.
- Any classification of the states of $\mathscr{W}$ provides a classification of the states of $\rho(\mathscr{W})''$, corresponding to the extendable states of $\rho(\mathscr{W})$.

[1] Fannes, M.; Verbeure, A.: *On the time evolution automorphisms of the CCR-algebra for quantum mechanics*. Commun. math. Phys. **35**, 257-264 (1974)

States

Let $\mathscr{A}$ be a unital C^*-algebra.

Definition 1 (State)

A linear functional ω on $\mathscr{A}$ is said to be *positive* if $\omega(a^*a) \geq 0$, $\forall a \in \mathscr{A}$. A positive, normalized linear functional is called a *state*, which is called *pure* if it cannot be written as a convex linear combination of other states.

Examples.

- Let $\mathscr{A} \subset \mathcal{B}(\mathcal{H})$ and $\Psi \in \mathcal{H}$. Then $\omega_\Psi(a) := \langle \Psi, a\,\Psi \rangle_{\mathcal{H}}$ is a state.
- Let X be a compact Hausdorff topological space, and $\mathscr{A} = C(X)$. Then any regular Borel measure μ on X defines a state via $\omega_\mu(a) := \int_X a(x)\,dx$. These are all the states.
- In the previous example, the states $\omega_{\delta_y}(a) = a(y)$ are the pure states of $C(X)$.

State Space

The set of states of $\mathscr{A}$ will be denoted with $\mathcal{S}_{\mathscr{A}}$. The subset of pure states is denoted with $\mathcal{P}_{\mathscr{A}}$.

The space $\mathcal{S}_{\mathscr{A}}$ is given the weak-$*$ topology. Net convergence is given by

$$\omega_i \to \omega \qquad \Longleftrightarrow \qquad \omega_i(a) \to \omega(a)\,, \quad \forall a \in \mathscr{A}\,.$$

With this, $\mathcal{S}_{\mathscr{A}}$ becomes a compact Hausdorff space.

The choice of the weak-$*$ topology has a physical basis: we want two states ω_1, ω_2 to be close together iff measuring any observable in these states yields similar values, *i.e.*, $\omega_1(a)$ is close to $\omega_2(a)$ for any $a \in \mathscr{A}$.

Equivalence of States

We need a topological notion of equivalence of states.

Definition 2 (Equivalence of states)

Given a subset $\mathcal{K} \subseteq \mathcal{S}_{\mathscr{A}}$ we will say that two states $\omega_0, \omega_1 \in \mathcal{K}$ are *equivalent* (*inside* $\mathcal{K}$) if there exists a continuous map $[0,1] \ni t \mapsto \omega_t \in \mathcal{K}$ joining ω_0 and ω_1. We denote the equivalence classes of $\mathcal{K}$ by $\Omega(\mathcal{K})$.

Physically, this condition is meant to reflect the absence of a phase change between both states.

Two states being equivalent on $\mathcal{K}$ does not equate to them being equivalent on bigger or smaller subspaces of $\mathcal{S}_{\mathscr{A}}$.

Weyl C*-algebra

The *Weyl algebra* (*CCR algebra*) $\mathscr{W}_0$ is the algebra generated by the elements $u_\alpha, v_\beta,\ \alpha, \beta \in \mathbb{R}^d$ with product laws given by:

$$u_\alpha v_\beta \;=\; \mathrm{e}^{\mathrm{i}\,\alpha\cdot\beta}\; v_\beta u_\alpha\ , \quad u_\alpha u_{\alpha'} \;=\; u_{\alpha+\alpha'}\ , \quad v_\beta v_{\beta'} \;=\; v_{\beta+\beta'}$$

We may define a *-involution by $u_\alpha^* = u_{-\alpha},\ v_\beta^* = v_{-\beta}$, and there is a unique norm $\|\cdot\|$ on $\mathscr{W}_0$ satisfying the C^* condition[2].

Taking the norm closure of the Weyl algebra, we obtain the *Weyl C*-algebra* $\mathscr{W}$.

[2] Manuceau, J.; Sirugue, M.; Testard, D.; Verbeure, A.: *The Smallest C^*-Algebra for Canonical Commutations Relations*

Regularity

Definition 3

A state $\omega \in \mathcal{S}_{\mathscr{W}}$ is called *regular* if both $\alpha \mapsto \omega(u_\alpha v_\beta)$ and $\beta \mapsto \omega(u_\alpha v_\beta)$ are continuous functions $\mathbb{R}^d \to \mathbb{C}$, and *semi-regular* if only one is.

Theorem 4 (Stone - Von Neumann)

All GNS representations of pure regular states of the Weyl C-algebra are unitarily equivalent to the Schrödinger representation.*

During this talk we will deal exclusively with β-regular states.

Translation symmetries

The Weyl C^*-algebra can be endowed with several groups of symmetries.

We will deal with symmetries given by the group of *space translations*, its (strongly continuous) action given by $\mathbb{R}^d \ni \lambda \mapsto \tau_\lambda \in \mathrm{Aut}(\mathscr{W})$,

$$\tau_\lambda(a) := v_\lambda a v_\lambda^* , \qquad a \in \mathscr{W} .$$

Specifically, in this talk we will focus on a discrete subgroup $\Gamma \subset \mathbb{R}^d$ of translations as our group of symmetries, implemented by τ_γ, $\gamma \in \Gamma$.

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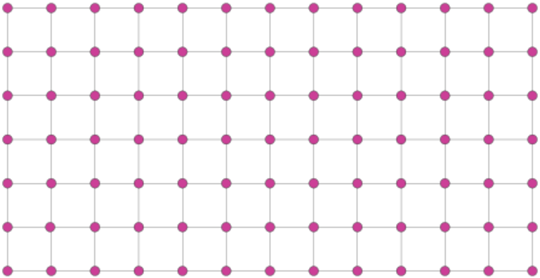
The symmetries: lattice translations

In our case, the symmetries consist of translations by a lattice $\Gamma \simeq \mathbb{Z}^d$ with l.i. basis vectors (i.e., group generators) $\{\mathfrak{e}^j\}_{j=1}^d \subset \mathbb{R}^d$, *i.e.*,

$$\Gamma := \{\gamma \in \mathbb{R}^d \mid \gamma := \gamma_1 \mathfrak{e}^1 + \ldots + \gamma_d \mathfrak{e}^d , \ \gamma_1, \ldots, \gamma_d \in \mathbb{Z}\} \simeq \mathbb{Z}^d .$$

Bidimensinal picturization (Wikimedia Commons):



Dual lattice and Brillouin zone

Γ has an associated *dual lattice*

$$\Gamma' \;:=\; \{\gamma' \in \mathbb{R}^d \mid \gamma' \cdot \gamma \in 2\pi\mathbb{Z}\,,\;\; \forall\, \gamma \in \Gamma\} \;\simeq\; \mathbb{Z}^d\;.$$

It coincides with the lattice generated by the dual basis $\{\mathfrak{f}^1, \ldots, \mathfrak{f}^d\}$ defined by $\mathfrak{f}^i \cdot \mathfrak{e}^j = 2\pi\delta_{i,j}$. With this, we define the *Brillouin zone*

$$\mathbb{B}_\Gamma \;:=\; \mathbb{R}^d/\Gamma' \;\simeq\; \mathbb{T}^d\;.$$

Γ' has an associated *unit cell*

$$Q_{\Gamma'} \;:= \{y \in \mathbb{R}^d \mid y = y_1\mathfrak{f}^1 + \ldots + y_d\mathfrak{f}^d\;,\;\; y_1, \ldots, y_d \in [0,1)\}\;,$$

There is a bijection between $Q_{\Gamma'}$ and $\mathbb{B}_\Gamma$.

The symmetries

We consider the action of Γ on $\mathscr{W}$ to be $\tau_\gamma(a) = v_\gamma a v_\gamma^*$.

Definition 5 (Lattice invariant states)

A state $\omega \in \mathcal{S}_{\mathscr{W}}$ is Γ-*invariant* (ΓI) if $\omega \circ \tau_\gamma = \omega$ for all $\gamma \in \Gamma$. The set of ΓI states will be denoted with $\mathcal{S}_{\mathscr{W}}^{\Gamma}$, and the subset of pure ΓI states with $\mathcal{P}_{\mathscr{W}}^{\Gamma} := \mathcal{S}_{\mathscr{W}}^{\Gamma} \cap \mathcal{P}_{\mathscr{W}}$.

We also define the space of pure, Γ-invariant, β-regular states:

$$\mathcal{P}_{\mathscr{W}}^{\Gamma,\beta} := \{\omega \in \mathcal{P}_{\mathscr{W}}^{\Gamma} \mid \omega \text{ is semi-regular in the parameter } \beta\}$$

Concrete representation

Let $\mathfrak{h}_\Gamma := L^2(\mathbb{R}^d/\Gamma, \mathrm{d}\nu)$, with $\mathrm{d}\nu$ the normalized Haar measure on $\mathbb{R}^d/\Gamma$.

Consider the two families of operators on $\mathfrak{h}_\Gamma$ defined by

$$\begin{aligned} (S_\beta f)(y) &:= f(y-\beta) \\ (F_{\gamma'} f)(y) &:= \mathrm{e}^{\mathrm{i}\,\gamma' \cdot y} f(y) \end{aligned} \tag{1}$$

for every $\gamma' \in \Gamma'$ and $\beta \in \mathbb{R}^d$. They satisfy

$$F_{\gamma'} S_\beta = \mathrm{e}^{\mathrm{i}\,\gamma' \cdot \beta} S_\beta F_{\gamma'}\,, \quad F_{\gamma'} F_{\eta'} = F_{\gamma'+\eta'}\,, \quad S_\beta S_\sigma = S_{\beta+\sigma}\,.$$

Lattice-invariant states

Let $\mathscr{G}_1(\mathfrak{h}_\Gamma)$ be the set of 1-dimensional projections of $\mathfrak{h}_\Gamma$.

Proposition 1 (Bloch-wave states [BMPS], Theorem 3.13)

Every element of $\mathcal{P}_{\mathscr{W}}^{\Gamma,\beta}$ is of the form

$$\omega_{(\kappa,P)}(u_\alpha v_\beta) \;:=\; \chi_{\Gamma'}(\alpha)\; \mathrm{e}^{-\mathrm{i}\,\kappa\cdot\beta}\,\mathrm{Tr}_{\mathfrak{h}_\Gamma}(PF_\alpha S_\beta) \tag{2}$$

where $(\kappa, P) \in Q_{\Gamma'} \times \mathscr{G}_1(\mathfrak{h}_\Gamma)$. This correspondence is bijective.

Covariance property

One cannot replace $\kappa \in Q_{\Gamma'}$ by $[\kappa] \in \mathbb{B}_{\Gamma}$:

$$\omega_{(\kappa+\gamma',P)} \neq \omega_{(\kappa,P)}$$

In fact, introducing the family of automorphisms

$$\lambda_{\gamma'}(A) := F_{\gamma'} A F_{\gamma'}^{*}\,, \qquad A \in \mathscr{B}(\mathfrak{h}_{\Gamma})\,,$$

one gets

$$\omega_{(\kappa+\gamma',P)} = \omega_{(\kappa,\lambda_{\gamma'}(P))}. \tag{3}$$

The Grassmann bundle

Let ${}^{\mathrm{w}}\mathscr{G}_1(\mathfrak{h}_\Gamma)$ be $\mathscr{G}_1(\mathfrak{h}_\Gamma)$ equipped with the WOT.

We endow $\mathbb{R}^d \times {}^{\mathrm{w}}\mathscr{G}_1(\mathfrak{h}_\Gamma)$ with the Γ'-action $(x, P) \mapsto (x + \gamma', \lambda_{-\gamma'}(P))$ and define the orbit space

$$\mathfrak{Gr}_1 \;:=\; \big(\mathbb{R}^d \times {}^{\mathrm{w}}\mathscr{G}_1(\mathfrak{h}_\Gamma)\big)/\Gamma'.$$

This space has the structure of a Grassmann bundle of rank 1, with base space $\mathbb{B}_\Gamma$ and typical fibre ${}^{\mathrm{w}}\mathscr{G}_1(\mathfrak{h}_\Gamma)$. It is in fact trivial:

$$\mathfrak{Gr}_1 \;\simeq\; \mathbb{B}_\Gamma \times {}^{\mathrm{w}}\mathscr{G}_1(\mathfrak{h}_\Gamma)$$

We denote the orbit of a point (x, P) by $[x, P]_{\Gamma'}$.

Topology of Γ-invariant states

Any $(k, P) \in \mathbb{R}^d \times \mathscr{G}_1(\mathfrak{h}_\Gamma)$ defines an element $\omega_{(k,P)} \in \mathcal{P}_{\mathscr{W}}^{\Gamma,\beta}$ via

$$\omega_{(k,P)}(u_\alpha v_\beta) \ := \ \chi_{\Gamma'}(\alpha) \ \mathrm{e}^{-\mathrm{i}\, k\cdot\beta}\, \mathrm{Tr}_{\mathfrak{h}_\Gamma}(P F_\alpha S_\beta) \ , \qquad (4)$$

and $\omega_{(k,P)} = \omega_{(\widetilde{k},\widetilde{P})} \iff [k, P]_{\Gamma'} = [\widetilde{k}, \widetilde{P}]_{\Gamma'}$ in $\mathfrak{Gr}_1$.

Theorem 6 (De Nittis - R., 2024)

The prescription (4) *provides an homeomorphism*

$$\Phi \ : \ \mathfrak{Gr}_1 \ \longrightarrow \ \mathcal{P}_{\mathscr{W}}^{\Gamma,\beta} \ .$$

As a consequence $\Omega(\mathcal{P}_{\mathscr{W}}^{\Gamma,\beta}) = \{[\omega_0]\}$ *is a singleton.*

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States from sections

Let $\pi : \mathfrak{Gr}_1 \to \mathbb{B}_\Gamma$ be the Grassmann bundle mentioned before. Denote by $\mathrm{Sec}(\mathfrak{Gr}_1)$ the set of *continuous sections* of $\mathfrak{Gr}_1$, *i.e.* the set of $F : \mathbb{B}_\Gamma \to \mathfrak{Gr}_1$ such that $(\pi \circ F)(\kappa) = \kappa$ for every $\kappa \in \mathbb{B}_\Gamma$.

Consider the homeomorphism $\Phi : \mathfrak{Gr}_1 \longrightarrow \mathcal{P}^{\Gamma,\beta}_{\mathscr{W}}$. For every $F \in \mathrm{Sec}(\mathfrak{Gr}_1)$ the composition $\Phi_F := \Phi \circ F$ is a continuous map

$$\Phi_F : \mathbb{B}_\Gamma \to \mathcal{P}^{\Gamma,\beta}_{\mathscr{W}}.$$

For every $\kappa \in \mathbb{B}_\Gamma$, $\omega_{F(\kappa)} := \Phi_F(\kappa)$ is an element of $\mathcal{P}^{\Gamma,\beta}_{\mathscr{W}}$.

Non-degenerated gapped states

We now define a topologically richer family of states.

Definition 7 (Non-degenerated gapped states)

Let $F \in \mathrm{Sec}(\mathfrak{Gr}_1)$ be a section, μ the normalized Haar measure on $\mathbb{B}_\Gamma$ and $\rho \in L^1(\mathbb{B}_\Gamma, \mu)$ a positive and L^1-normalized function. The state

$$\omega_{F,\rho} := \int_{\mathbb{B}_\Gamma} \mathrm{d}\mu(\kappa)\, \rho(\kappa)\omega_{F(\kappa)}$$

is called the non-degenerated gapped state with section F and distribution ρ. The set of non-degenerated gapped states is denoted with $\mathcal{G}^\Gamma_{\mathscr{W},1}$.

Topological content of non-degenerated gapped states

Let $[X, Y]$ denote the homotopy equivalence classes of functions $X \to Y$.

Lemma 8 (modulo one detail left to prove)

There is a bijection

$$\Omega(\mathcal{G}^{\Gamma}_{\mathscr{W},1}) \simeq [\mathbb{B}_{\Gamma}, \mathfrak{Gr}_1] .$$

Let $H^n(X, \mathbb{Z})$ denote the n-th cohomology group of the space X with integer coefficients.

Theorem 9 (De Nittis - R., 2024)

There are bijective correspondences

$$\Omega(\mathcal{G}^{\Gamma}_{\mathscr{W},1}) \simeq H^2(\mathbb{B}_{\Gamma}, \mathbb{Z}) \simeq \begin{cases} 0 & \text{if } d = 1 \\ \mathbb{Z}^{\oplus \binom{d}{2}} & \text{if } d \geqslant 2 . \end{cases}$$

Relation with vector bundles and K-theory

From the proof of the last Theorem, it also follows that

$$\Omega(\mathcal{G}^{\Gamma}_{\mathscr{W},1}) \simeq \mathrm{Vec}^1_{\mathbb{C}}(\mathbb{B}_{\Gamma})$$

where $\mathrm{Vec}^1_{\mathbb{C}}(X)$ denotes the set of isomorphism classes of complex line bundles over X.

In this sense the bijection between $\Omega(\mathcal{G}^{\Gamma}_{\mathscr{W},1})$ and $H^2(\mathbb{B}_{\Gamma}, \mathbb{Z})$ described in Theorem 9 can be thought in terms of *Chern classes* as in the standard theory of classification of line bundles:

$$\mathrm{Vec}^1_{\mathbb{C}}(\mathbb{B}_{\Gamma}) \simeq [\mathbb{B}_{\Gamma}, K(\mathbb{Z}, 2)] \overset{c_1}{\simeq} H^2(\mathbb{B}_{\Gamma}, \mathbb{Z})$$

where the map c_1 which provides the bijection (indeed a group isomorphism) is known as the *first Chern class*. Moreover, if $1 \leq d \leq 3$,

$$\Omega(\mathcal{G}^{\Gamma}_{\mathscr{W},1}) \simeq \widetilde{K}^0(\mathbb{B}_{\Gamma}).$$

Generic gapped states

Let $\mathfrak{Gr}_N$ be the Grassmann bundle of rank N obtained just as $\mathfrak{Gr}_1$ by replacing ${}^{\mathrm{w}}\mathscr{G}_1(\mathfrak{h}_\Gamma)$ with ${}^{\mathrm{w}}\mathscr{G}_N(\mathfrak{h}_\Gamma)$.

Definition 10 (Degenerated gapped states)

Let $F \in \mathrm{Sec}(\mathfrak{Gr}_N)$ be a section and $\rho \in L^1(\mathbb{B}_\Gamma, \mu)$ a positive and L^1-normalized function. The state defined by

$$\omega_{F,\rho} := \int_{\mathbb{B}_\Gamma} \mathrm{d}\mu(\kappa)\, \frac{\rho(\kappa)}{N} \omega_{F(\kappa)}$$

will be called the N-degenerated gapped state with section F and distribution ρ. The set of these states will be denoted with $\mathcal{G}^\Gamma_{\mathscr{W},N}$.

Topology of gapped states

Theorem 11 (De Nittis - R., 2024)

There is a bijective correspondence

$$\Omega(\mathcal{G}^{\Gamma}_{\mathscr{W},N}) \simeq \mathrm{Vec}^{N}_{\mathbb{C}}(\mathbb{B}_{\Gamma}) \, .$$

K-theoretical classification in low dimension

Let $N \in \mathbb{N}$. When $1 \leq d \leq 3$, a standard result in the theory of vector bundles provides $\mathrm{Vec}^N_{\mathbb{C}}(\mathbb{B}_\Gamma) \simeq \mathrm{Vec}^1_{\mathbb{C}}(\mathbb{B}_\Gamma)$. As a consequence,

$$\Omega(\mathcal{G}^\Gamma_{\mathscr{W},N}) \;\simeq\; \Omega(\mathcal{G}^\Gamma_{\mathscr{W},1}) \;, \qquad \text{if } \; 1 \leq d \leq 3 \;.$$

For $d = 4$ one knows that $\mathrm{Vec}^N_{\mathbb{C}}(\mathbb{B}_\Gamma)$ is described by the second and fourth cohomology groups. Therefore

$$\Omega(\mathcal{G}^\Gamma_{\mathscr{W},N}) \;\simeq\; H^2(\mathbb{B}_\Gamma, \mathbb{Z}) \oplus H^4(\mathbb{B}_\Gamma, \mathbb{Z}) \;\simeq\; \mathbb{Z}^{\oplus 7} \;, \qquad \text{if } \; d = 4 \;.$$

In both cases one gets

$$\Omega(\mathcal{G}^\Gamma_{\mathscr{W},N}) \;\simeq\; \widetilde{K}^0(\mathbb{B}_\Gamma) \;, \qquad \text{if } \; 1 \leq d \leq 4 \;,$$

showing the generality of K-theory in the classification of gapped states.

Section

What we have obtained

- We defined a notion of topological equivalence of states, and showed the topological triviality of many classes of them.
- Using this notion we managed to recover the topological structures present in gapped states, such as their relation with complex vector bundles over the Brillouin zone $\mathbb{B}_\Gamma$.
- We showed that the reduced K-theory $\widetilde{K}^0(\mathbb{B}_\Gamma)$ classifies the gapped states in low dimension.
- Finally, throughout this work we set the framework for an even more general topological classification of states.

What is next? The ideas

- Thermal states: Having classified the gapped states which are parametrized by $\mathfrak{Gr}_N := \left(\mathbb{R}^d \times {}^{\mathrm{w}}\mathscr{G}_N(\mathfrak{h}_\Gamma)\right)/\Gamma'$, we may classify families of states where ${}^{\mathrm{w}}\mathscr{G}_N(\mathfrak{h}_\Gamma)$ is replaced by an appropriate set of trace-class operators.

- Non-regular Γ-invariant states: We focused our discussion in the case of β-regular states, but there is plenty more than these, albeit more pathological. We have succeeded on obtaining and classifying other classes of interesting states, but there is much more to be done.

- Infinite degrees of freedom? Topological invariants in GNS representations? Many more angles to explore!

Thanks!

Thank you for your attention!

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