Edge modes at soft walls.

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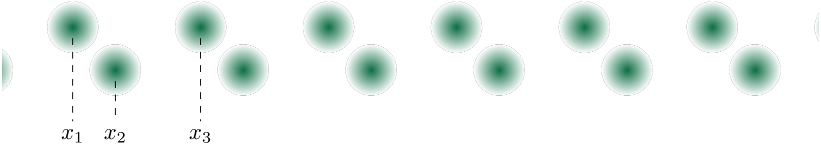
Joint work with C. Gomez, D. Gontier

Outline

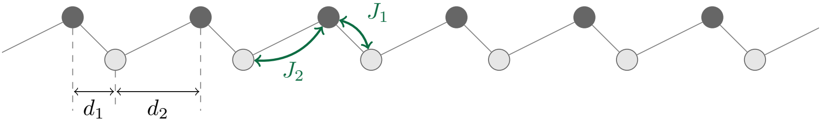
- Bulk models, band structure, Bloch transforms.
- Edges and soft walls.
- The spectral flow argument.
- Results for the Wallace model.

The Su–Schrieffer–Heeger (SSH) model - tight binding



Continuum model:

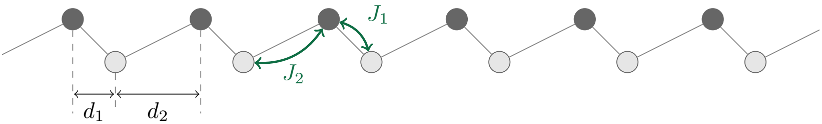
$$H = -\Delta + \sum_{k \in \mathbb{Z}} V(. - x_k), \quad \mathcal{H} = L^2(\mathbb{R}^3)$$



Tight-Binding model:

$$H = J_2 \sum_{j \text{ even}} |j\rangle\langle j+1| + J_1 \sum_{j \text{ odd}} |j\rangle\langle j+1| + \text{h. c. }, \quad \mathcal{H} = \ell^2(\mathbb{Z})$$

The SSH model - periodic version

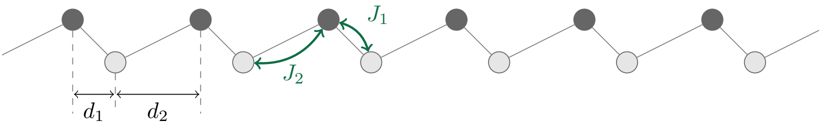


Take a unit cell with two atoms: $\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2)$

$$(\widetilde{H}\Psi)_n := a_{n-1}^* \Psi_{n-1} + b_n \Psi_n + a_n \Psi_{n+1},$$

with $b = \begin{pmatrix} 0 & J_1 \\ J_1^* & 0 \end{pmatrix}$ and $a = \begin{pmatrix} 0 & 0 \\ J_2 & 0 \end{pmatrix}$.

The SSH model -Bloch transform



Any $\Psi \in \mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2)$ can be seen as Fourier coefficient of a periodic function

$$\left(\mathcal{F}[\Psi]\right)(k) := \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} \Psi_m e^{-ikm} \in L^2([-\pi, \pi), \mathbb{C}^2).$$

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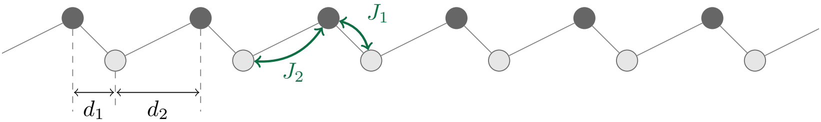

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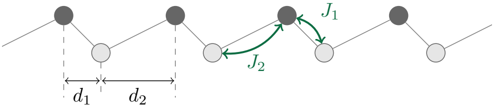
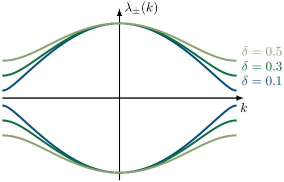
And $\mathcal{F}$ is a unitary transformation that diagonalizes $\widetilde{H}$

$$\left(\mathcal{F}\widetilde{H}\mathcal{F}^* u\right)(k) = \widetilde{H}_k u(k), \quad \text{with} \qquad \widetilde{H}_k := a^* e^{-ik} + b + a e^{ik}.$$

The SSH model - Bulk Spectrum





Explicitely,

$$\widetilde{H}_k = b + a e^{ik} + a^* e^{-ik} = \begin{pmatrix} 0 & J_1 + J_2 e^{-ik} \\ J_1 + J_2 e^{ik} & 0 \end{pmatrix}.$$

Bulk bands in general

Generalizations

- More atoms per unit cell $\to \ell^2(\mathbb{Z}, \mathbb{C}^N)$
- Include magnetic fields $\to$ complex hopping coefficients
- Interactions between second neighbor cells and beyond
- Dimension $2 \to \ell^2(\mathbb{Z}^2, \mathbb{C}^N)$

In all cases

- There are N bands
- They correspond to essential spectrum
- Describe transport and dispersion in the bulk material

Edge modes

Half-line : H en $\ell^2(\mathbb{N}, \mathbb{C}^2)$: solve $H\Psi = 0$

$$H = \begin{pmatrix} 0 & J_1 & & & \\ J_1 & 0 & J_2 & & \\ & J_2 & 0 & J_1 & \\ & & J_1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

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$$\begin{cases} \Psi_{2n} = 0 \\ \Psi_{2n+1} = (-J_1/J_2)^n \Psi_1 \end{cases}$$

Edge modes

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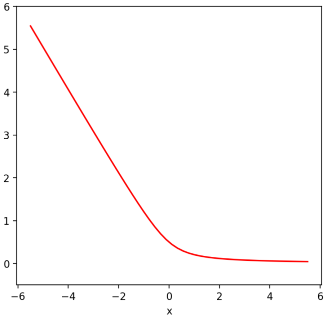
Edge modes at zero energy...Majorana fermions ?

Soft walls

Study $H_t^\sharp := \widetilde{H} + W_t$ en $\ell^2(\mathbb{Z}, \mathbb{C}^2)$:



f is Lipschitz if
$|f(x) - f(y)| \leq L|x - y|$

Wall potential

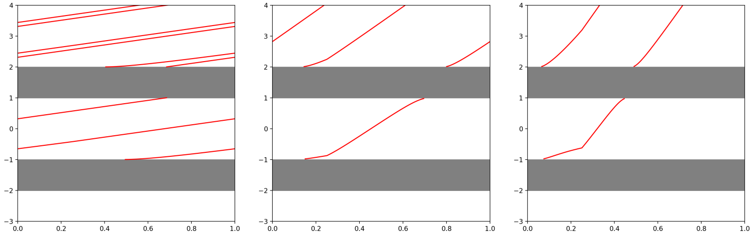
$w : \mathbb{R} \mapsto \mathbb{R}$ is

- Lipschitz-continuous
- As $x \to -\infty$, $w(x) \to +\infty$
- As $x \to +\infty$, $w(x) \to 0$

$$W_t(n) := \begin{pmatrix} w(n - t - x_1) & 0 \\ 0 & w(n - t - x_2) \end{pmatrix}$$

Edge modes

Plot $\sigma(H^\sharp(t))$: numerics with $w = -\nu[x]_-$ for $\nu = 1, 5, 10$.



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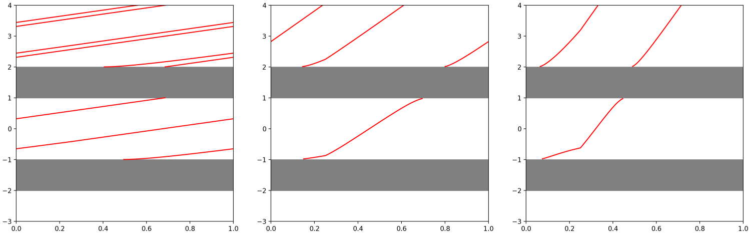

Observations

- $t \mapsto \sigma(H^\sharp(t))$ is periodic

Edge modes

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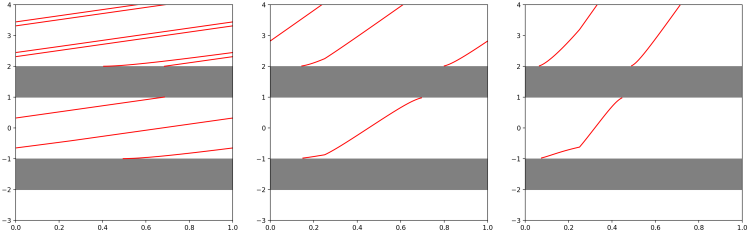

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Edge modes

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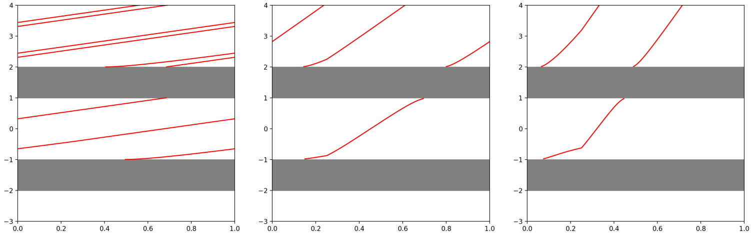
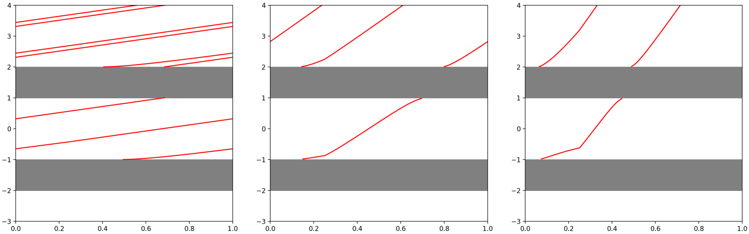


Observations

- $t \mapsto \sigma(H^\sharp(t))$ is periodic
- $\sigma_{\text{ess}}(H^\sharp(t)) = \sigma_{\text{bulk}}$
- Edge modes depend on ν

Edge modes

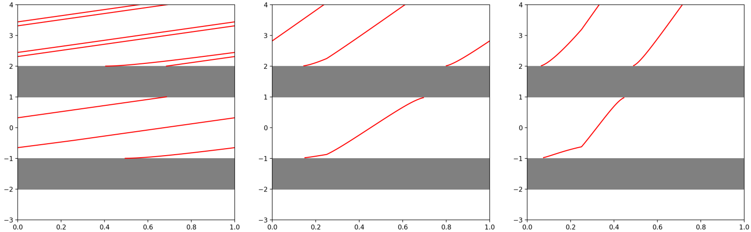
Plot $\sigma(H^\sharp(t))$: numerics for $J_1 = 3/2$, $J_2 = 1/2$, with $w = -\nu[x]_-$ for $\nu = 1, 5, 10$.

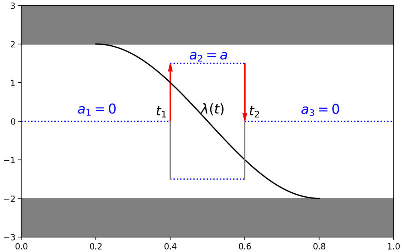


Theorem

If w is Lipschitz with constant L and there is a gap above the $\mathcal{N}$-th band of width $\ell > L$, then $\sigma(H_t)$ has at least $\mathcal{N}\lfloor l/L \rfloor$ eigenvalues in this gap.

Spectral flow

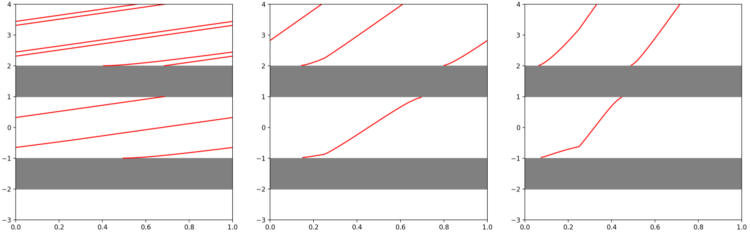




Definition

$\mathrm{Sf}(A_t, E, [0,1])$ = the number of eigenvalues branches that cross E downwards as t increases from 0 to 1.

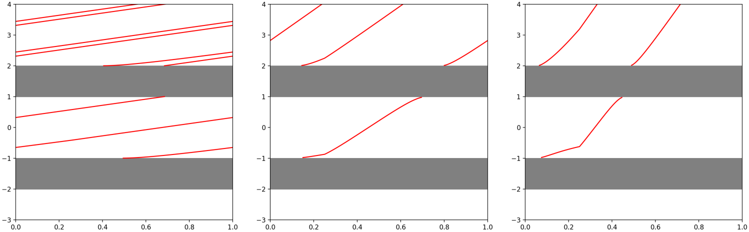
Spectral flow



Theorem

For E in a bulk gap, such that there are $\mathcal{N}(E)$ bands below E, we have $\mathrm{Sf}(H_t^\sharp, E, [0,1]) = -\mathcal{N}(E)$.

Spectral flow



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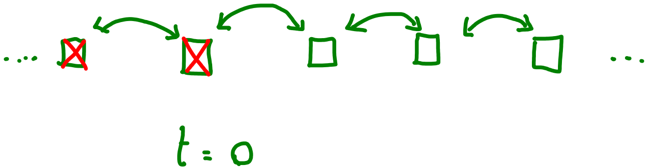

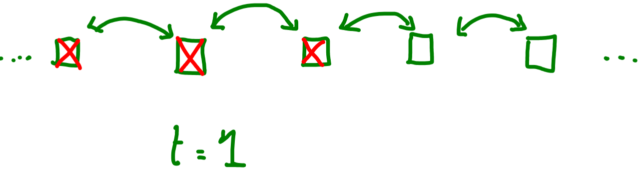

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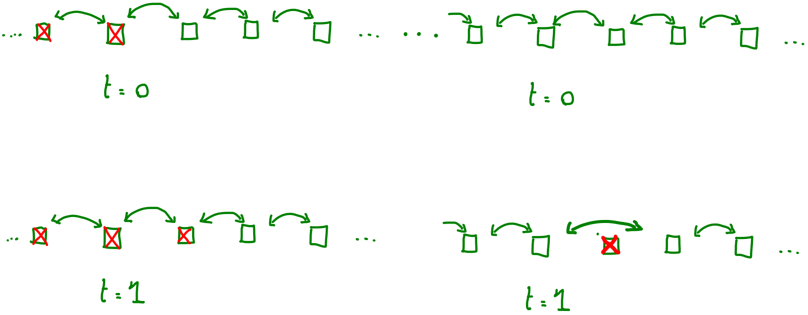
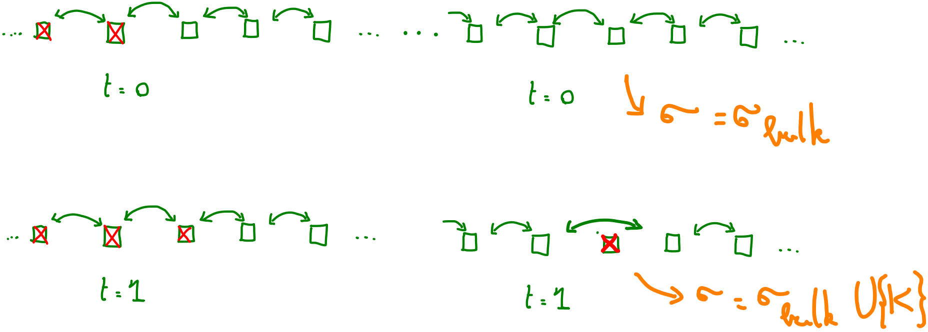


Steep-Wall-model

Dislocated model

Sketch of the proof



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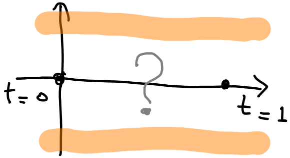
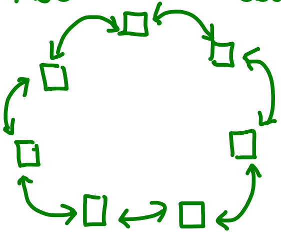
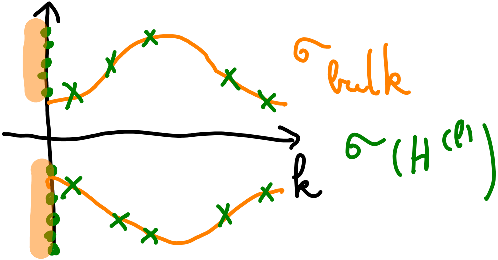


Sketch of the proof

Periodic Model $2\ell+1$ sites



$$\sigma(H^{(\ell)}) = \bigcup_{j=-\ell}^{\ell} \sigma(H_{k_j})$$



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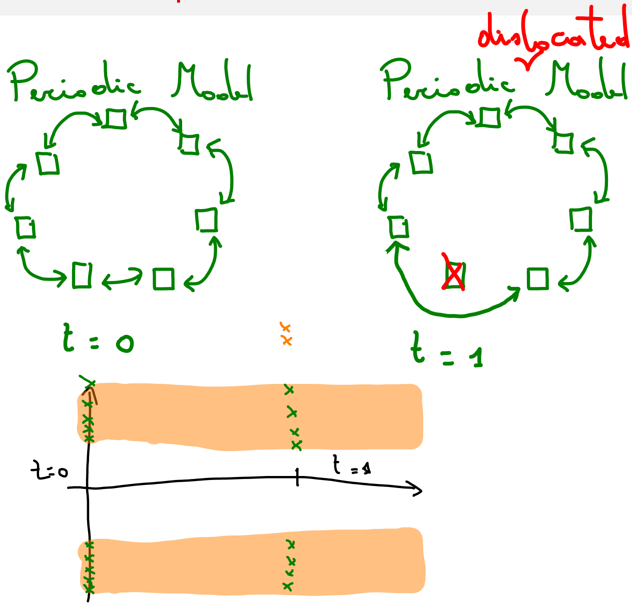

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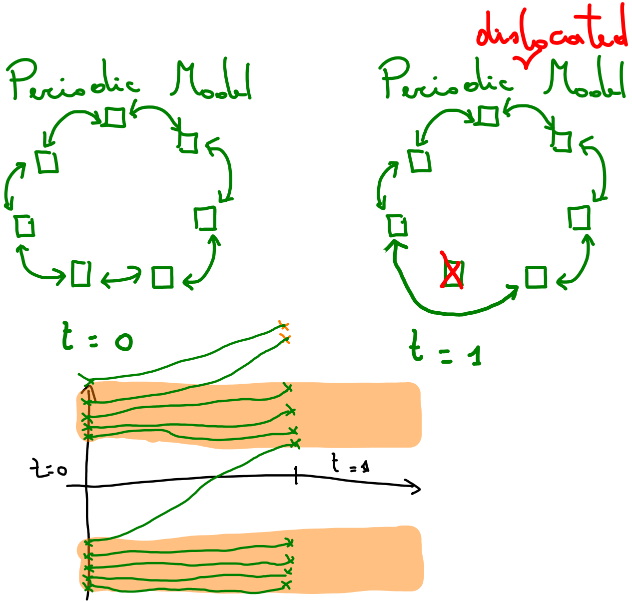
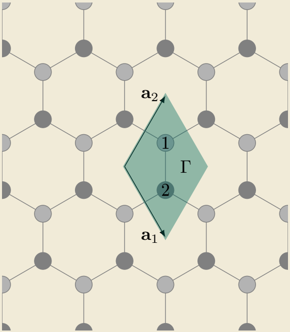
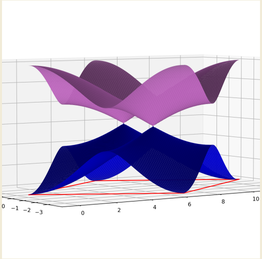


For two-dimensional materials: The Wallace model

Honeycomb Lattice

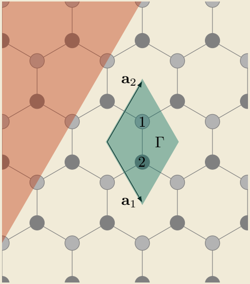


Band structure

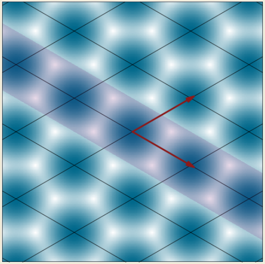


The Wallace model with a zigzag wall

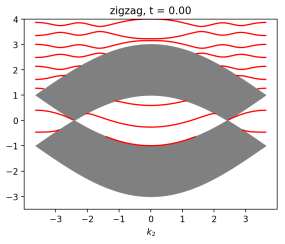
Lattice picture



Reciprocal space

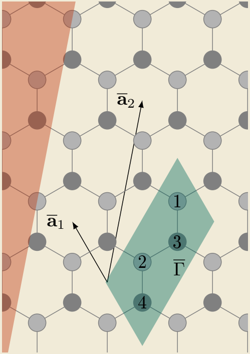


Edge states



The Wallace model with a rational cut

Lattice picture



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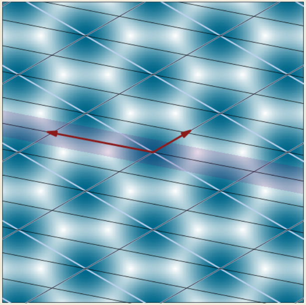

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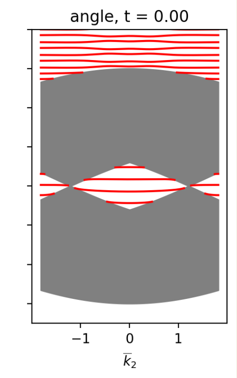

General conclusions

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- A slowly varying potential gives more edge states

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- A slowly varying potential gives more edge states
- For a wall in $d = 2$ parallel to $n\mathbf{a}_1 + m\mathbf{a}_2$, the gaps fill as n, m increase

Muchas gracias !