Quantum Systems at Finite Temperature $T > 0$

Berry and Uhlmann Phases.

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Content

The Ising XY model

Let us consider the Ising XY chain of n sites with closed boundary conditions. Let us start with a spin chain of $n = 2L$ sites, say $\Lambda \subseteq \mathbb{Z}$. For each point x in the lattice Λ, we associate the complex Hilbert space $\mathbb{C}^2$ and the algebra $\mathcal{A}_x = M_2(\mathbb{C})$. Thus, the whole algebra in the lattice corresponds to

$$\mathcal{A}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{A}_x .$$

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$$\mathcal{A}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{A}_x .$$

The Hamiltonian operator for this physical system is given by:

$$H = -\frac{1}{2} \sum_{j=-L}^{L-1} \left(\frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y + \lambda \sigma_j^z \right),$$

CAR algebra

Definition

Let $\mathcal{H}$ be a complex Hilbert space. The Fock space of $\mathcal{H}$ is defined as the Hilbert space direct sum

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n \geq 0} \bigwedge^n \mathcal{H},$$

where $\bigwedge^0 \mathcal{H} := \mathbb{C}$ by convention.

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Given $\xi \in \mathcal{H}$, consider the linear operator:

$$a^*(\xi) : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}), \quad \eta_1 \wedge \ldots \wedge \eta_n \mapsto \xi \wedge \eta_1 \wedge \ldots \wedge \eta_n.$$

This operator satisfies $||a^*(\xi)|| = ||\xi||$. It is known as *creation operator*, and its adjoint, $a(\xi)$, as *annihilation* operator.

The Ising XY model transformed

The Hamiltonian operator describing the Ising model can be written in terms of elements of the CAR algebra as:

$$H = -\frac{1}{2} \sum_{i=1}^{n} \left[\gamma (a_i^* a_{i+1}^* - a_i a_{i+1}) + (a_i^* a_{i+1} + a_{i+1}^* a_i) - 2\lambda a_i^* a_i + \lambda \right],$$

where this operator acts in the Fock space of $\mathcal{H}_\Lambda := l^2(\Lambda)$, with $\Lambda = \{1 \ldots n\}$, and it has been imposed boundary conditions such that $a_{n+1} = a_1$.

The Ising XY model transformed

We define the discrete Fourier transform $U_F : \mathcal{H}_\Lambda \to \tilde{\mathcal{H}}_\Lambda$ as:

$$\tilde{f}(k) = [U_F \ f](k) = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} f_l e^{-i\varphi_k l}.$$

Here, $\tilde{\mathcal{H}}_\Lambda$ denotes the dual lattice. When $\Lambda = \mathbb{Z}$, then $\varphi_k \to \varphi \in [0, 2\pi]$, and hence $\tilde{\mathcal{H}}_\Lambda = L^2([0, 2\pi], d\mu)$, with $d\mu(k) := \frac{1}{2\pi} d\varphi$.

The Ising XY model transformed

Consider the following operators d_k:

$$d_k := \frac{1}{\sqrt{n}} \sum_{l=1}^{n} a_l e^{-i\varphi_k l}.$$

These new operators fulfill the CAR relations, and it is not hard to see that the inverse is given by:

$$a_j = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{i\varphi_k j} d_k.$$

The Ising XY model transformed

Now, we can consider the following *self-dual Hilbert space*: $\tilde{\mathcal{K}}_\Lambda := \tilde{\mathcal{H}}_\Lambda \oplus \overline{\tilde{\mathcal{H}}}_\lambda$. Then, we can write the Hamiltonian in its bilinear form:

$$H = \sum_{k=1}^{n} h(k)\tilde{B}^*(k)\tilde{B}(k),$$

where $\tilde{B}(k) := d_k + d^*_{-k}$. More explicitly:

$$H = \frac{1}{2}\sum_{k=1}^{n} \begin{pmatrix} d_k^* & d_{-k} \end{pmatrix} \begin{pmatrix} (\cos\varphi_k - \lambda) & -i\gamma\sin\varphi_k \\ i\gamma\sin\varphi_k & -(\cos\varphi_k - \lambda) \end{pmatrix} \begin{pmatrix} d_k \\ d^*_{-k} \end{pmatrix}.$$

The Ising XY model transformed

The new one-particle Hamiltonian as an operator on $\tilde{K}_\Lambda$ can be expressed as:

$$\hat{h} = \begin{pmatrix} \hat{h}_1 & 0 & \cdots & 0 \\ 0 & \hat{h}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \hat{h}_n \end{pmatrix}, \quad \hat{h}_k = \begin{pmatrix} M_k & N_k \\ -N_k & M_k \end{pmatrix}.$$

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The unitary element U which diagonalizes this Hamiltonian satisfies:

$$\det\, U = \mathrm{sgn}\,(1 - \lambda).$$

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In the physics literature this is known as a topological $\mathbb{Z}_2$ index. There is a good mathematical development behind this, which unfortunately is beyond the scope of this talk.

Berry's phase

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Let $\mathcal{A}$ be a unital C^*-algebra, and let us denote by Q the orbit of a fixed idempotent element by the adjoint action of the invertible elements:

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Proposition

[a] *Let $\pi : G \to Q$ be the continuous map given by $\pi(g) = gqg^{-1}$. This is a principal bundle with structural group*

$$G_0 := \{g \in G : \; gq = qg\}.$$

[a] Corach, Porta, and Recht, "Differential geometry of systems of projections in Banach algebras".

Berry's phase

On this bundle we have have the section

$$\sigma : Q \to G$$

given by:

$$\sigma(q') = q'q + (\mathbf{1} - q')(\mathbf{1} - q). \tag{1}$$

Berry's phase

On this bundle we can explore a geometry through the decomposition into vertical and horizontal subspaces in the following way:

$$V_g = g \cdot \left\{ qXq + q^{\perp}Xq^{\perp} : \ X \in \mathcal{A} \right\}, H_g = g \cdot \left\{ q^{\perp}Xq + qXq^{\perp} : \ X \in \mathcal{A} \right\}.$$

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Proposition

Let $\gamma : [0,1] \to Q$ be a smooth path. Then the horizontal lifting $\Gamma : [0,1] \to G$ with $\Gamma(0) = \mathbf{1}$ is the unique solution of the differential equation:

$$\dot{\Gamma} = \dot{\gamma}(2\gamma - \mathbf{1})\Gamma = [\dot{\gamma}, \gamma]\Gamma, \qquad \Gamma(0) = \mathbf{1}.$$

Berry's phase

Let us return to the Ising model. Consider the curve of self-adjoint operators $h : \mathbb{S}^1 \to M_2(\mathbb{C})$:

$$h(k) = \begin{pmatrix} (\cos k - \lambda) & -i\gamma \sin k \\ i\gamma \sin k & -(\cos k - \lambda) \end{pmatrix}.$$

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Let $P(k)$ be the curve of spectral projections onto the ground state space. By taking the horizontal lift as described earlier, we obtain a non-trivial holonomy, which corresponds precisely to

$$\mathrm{sgn}\ (1 - \lambda).$$

Self-dual algebra

Definition

Let $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space of even or infinite dimension, and let Γ be a complex conjugation on $\mathcal{K}$. Define the self-dual CAR algebra (sdCAR), denoted as $\mathcal{A}_{SD}(\mathcal{K},\Gamma)$ as the unital C^*-algebra generated by the symbols $B(f)$, with $f \in \mathcal{K}$, such that $B(f)^* := B(\Gamma f)$, and the self-dual CAR relation:

$$\{B(f), B(g)^*\} = \langle f, g\rangle \mathbf{1}.$$

Here, the map $f \mapsto B(f)$ is complex linear.
In addition, we say that an orthogonal projection $E : \mathcal{K} \to \mathcal{K}$ is a basis projection if $E + \Gamma E \Gamma = \mathbf{1}$.

Quasi-free states on self-dual algebras

Let $S \in \mathcal{B}(\mathcal{K})$ such that:

$$0 \leq S^* = S \leq \mathbf{1} \text{ and}$$
$$S + \Gamma S \Gamma = \mathbf{1}.$$

[1] Huzihiro Araki. "On Quasifree States of CAR and Bogoliubov Automorphisms". In: *Publications of The Research Institute for Mathematical Sciences* 6 (1970), pp. 385–442.

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This positive operator defines a state $\omega : \mathcal{A}_{SD}(\mathcal{K}, \Gamma) \to \mathbb{C}$, which as a two-site function corresponds to:

$$\omega(B(\psi)^* B(\varphi)) = \langle \psi, S\varphi \rangle.$$

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This family of states are known as quasi-free states.The positive operator S is known as the symbol of the state.
If $S = E$, a basis projection, then the state is pure.[1]

[1] Araki, "On Quasifree States of CAR and Bogoliubov Automorphisms".

Bilinear Hamiltonian

Definition

Let $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ be a complex Hilbert space, and let Γ be a complex conjugation Let $S \in \mathcal{B}(\mathcal{K})$ be a trace-class operator, and let $\{\psi_i\}$ be an orthonormal basis for $\mathcal{K}$. We define the bilinear element $B^*SB =: b(S)$ on $\mathcal{A}_{SD}(\mathcal{K}, \Gamma)$ as:

$$b(S) = \sum_{i,j} \langle \psi_i, S\psi_j \rangle B^*(\psi_j) B(\psi_i).$$

Notice that $b(S^*) = b(S)^*$. If S is self-adjoint and **self-dual**, i.e., $\Gamma S \Gamma = -S$, we called the bilinear element $b(S)$ a **bilinear Hamiltonian**.

Gibbs states and Equilibrium

Let us now state one of the most important propositions in this context, which builds upon various foundational concepts including Gibbs states and KMS states.

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Proposition

[a] *For any $\beta > 0$ and any self-adjoint, self-dual Hamiltonian $h \in \mathcal{K}$, the positive operator $S_\beta \in \mathcal{B}(\mathcal{K})$ defined by:*

$$S_\beta := \left[\mathbf{1} + e^{-\beta h}\right]^{-1},$$

is the symbol of ω_β satisfying:

$$\omega_\beta(B) = \frac{Tr\ \left(Be^{-\frac{\beta}{2}H}\right)}{Tr\ \left(e^{-\frac{\beta}{2}H}\right)}, \text{ where } H = b(h).$$

[a] Sequera, "Geometric and Topological aspects of quasi-free states on self-dual algebras".

Purification

Let us now consider the construction of a pure state from a symbol S.

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Consider the Hilbert space $\hat{\mathcal{K}} := \mathcal{K} \oplus \mathcal{K}$ and the orthogonal projection P_S on $\hat{\mathcal{K}}$ as:

$$P_S := \begin{pmatrix} S & S^{\frac{1}{2}} (\mathbf{1} - S)^{\frac{1}{2}} \\ S^{\frac{1}{2}} (\mathbf{1} - S)^{\frac{1}{2}} & \mathbf{1} - S \end{pmatrix}. \qquad (2)$$

On $\hat{\mathcal{K}}$ define the conjugation $\hat{\Gamma} := \Gamma \oplus (-\Gamma)$. It follows that P_S is a basis projection on $(\hat{\mathcal{H}}, \hat{\Gamma})$.

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On $\hat{\mathcal{K}}$ define the conjugation $\hat{\Gamma} := \Gamma \oplus (-\Gamma)$. It follows that P_S is a basis projection on $(\hat{\mathcal{H}}, \hat{\Gamma})$.

Proposition

[a] *The restriction of the quasi-free pure state ω_{P_S} to the self-dual algebra $\mathcal{A}_{SD}(\mathcal{K}, \Gamma)$ is the quasi-free state ω_S.*

[a] Araki, "On Quasifree States of CAR and Bogoliubov Automorphisms".

$T > 0$, C^*-algebras and Uhlmann Phases

Let $\mathcal{A}$ be a unital C^*-algebra faithfully represented on a separable Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Let G^+ be the set of positive elements of the C^*-algebra $\mathcal{A}$:

$$G^+ = \{b \in G : \; b^* = b \text{ and } b > 0\},$$

where G denotes, as usual, the space of invertible elements of the algebra $\mathcal{A}$.

$T > 0$, C^*-algebras and Uhlmann Phases

Each element $a \in G^+$ defines an inner product on the Hilbert space $\mathcal{H}$, say $\langle \cdot, \cdot \rangle_b$, given by:

$$\langle u, v \rangle_b := \langle bu, v \rangle.$$

This inner product defines an adjoint operator $*_b : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which can be related to the usual adjoint $*$ as:

$$T^{*_b} = b^{-1} T^* b.$$

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With this adjoint, we introduce the spaces of $b-$hermitian $\mathcal{A}_h^b$, $b-$antihermitian $\mathcal{A}_{ah}^b$, and the $b-$unitary elements $\mathcal{U}^b$ as:

$$\mathcal{A}_h^b = \{X \in \mathcal{A} : \ X = b^{-1} X^* b\}, \quad \mathcal{A}_{ah}^b = \{X \in \mathcal{A} : \ X = -b^{-1} X^* b\},$$

$$\mathcal{U}^b = \{u \in \mathcal{A} : \ uu^{*_b} = u^{*_b} u = \mathbf{1}\}.$$

$T > 0$, C^*-algebras and Uhlmann Phases

Fix an element $a \in G^+$, and let $\pi_a : G \to G^+$ the map given by:

$$\pi_a(g) = gag^*.$$

This defines a $\mathcal{U}^{a^{-1}}$-principal bundle $\pi_a : G \to G^+$.

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Definition

We define the Uhlmann's connection $1-$form

$$\omega : TG \to \text{Lie}\ (\mathcal{U}^{a^{-1}}),$$

for the $\mathcal{U}^{a^{-1}}-$principal bundle $\pi_a : G \to G^+$ as the $1-$form given by:

$$\omega_g(X) = g^{-1}X - g^{-1}Wg,$$

where W is the unique solution to the equation:

$$Wgag^*a + gag^*aW = Xag^*a + gaX^*a.$$

$T > 0$, C^*-algebras and Uhlmann Phases

On this principal bundle with the previous connection ω, we have how the horizontal lift is performed:[2]

[2] Gustavo Corach and Alejandra L. Maestripieri. "Geometry of positive operators and uhlmann's approach to the geometric phase". In: *Reports on Mathematical Physics* 47.2 (2001), pp. 287–299.

$T > 0$, C^*-algebras and Uhlmann Phases

On this principal bundle with the previous connection ω, we have how the horizontal lift is performed:[2]

Theorem

Let $\gamma : [0,1] \to G^+$ be a smooth curve such that $\gamma(0) = b$, and let $g \in G$ such that $\pi_a(g) = b$. Then, there is a unique horizontal lift $\Gamma : [0,1] \to G$ such that $\Gamma(0) = g$, namely the solution of the differential equation:

$$\gamma(t)\ a\ \dot{\Gamma}(t)\Gamma^{-1}(t) + \dot{\Gamma}(t)\Gamma^{-1}(t)\gamma(t)\ a = \dot{\gamma}(t)\ a, \qquad \Gamma(0) = g.$$

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Holonomies

With these constructions, we have two different holonomies for the curve

$$k \to S_\beta(k)$$

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$$k \to S_\beta(k)$$

of positive elements.

Together with Reyes and Sequera, we have shown that the spectra of both holonomies for these families of operators (the positive operators S_β and the projections P_β) coincide.

Holonomies

Proposition

Let $h : \mathbb{S}^1 \to \mathcal{A}$ be a smooth path of self-adjoint elements in a unital C^*-algebra $\mathcal{A}$. Consider the path $X_\beta(t) = e^{-\beta h(t)/2}$, and define the positive symbols:

$$S_\beta(t) = \left(\mathbf{1} + X_\beta(t)^2\right)^{-1}.$$

Let $P_{S_\beta}(t)$ be the family of orthogonal projections in $M_2(\mathcal{A})$ obtained from the purification 2. Then, if

$$[\dot{X}_\beta(t), S_\beta(t)X_\beta(t)] = 0,$$

the spectra of the holonomies resulting from the horizontal lifting of the path S_β in the Uhlmann fibration and the path P_{S_β} in the Berry fibration are equal.

Holonomies

Sketch of the proof:

Using the section σ in equation 1 and the horizontal lifting equation for Uhlmann, we get the equations:

$$\text{Uhlmann:} \qquad S\dot{\Gamma}\Gamma^{-1} + \dot{\Gamma}\Gamma^{-1}S = \dot{S}, \quad \Gamma(0) = \mathbf{1},$$

$$\text{Berry:} \qquad \dot{\Sigma} = -(\dot{S} + S^{-1}A\dot{A})\Sigma, \quad \Sigma(0) = \mathbf{1}$$

where $A = Se^{-\beta h/2}$.

Holonomies

Sketch of the proof:

Let Z be the solution to the Sylvester equation:

$$SZ + ZS = \dot{S}.$$

Hence, if Γ is the solution to the differential equation $\dot{\Gamma} = Z\Gamma$, then $(\Gamma^{-1})^*$ is the solution to the equation:

$$\dot{\Sigma} = -Z^*\Sigma.$$

Holonomies

Sketch of the proof:

Let Z be the solution to the Sylvester equation:

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Hence, if Γ is the solution to the differential equation $\dot{\Gamma} = Z\Gamma$, then $(\Gamma^{-1})^*$ is the solution to the equation:

$$\dot{\Sigma} = -Z^*\Sigma.$$

If we denote by Y the right-hand side of the Berry equation above, that is:

$$Y = \dot{S} + S^{-1}A\dot{A},$$

then if Y^* solves the Sylvester equation, we can conclude the equality of both spectra.

Holonomies

Sketch of the proof:

Let $X = e^{-\beta h/2}$. Then:

$$\{S, Y^*\} = \dot{S} + S[\dot{X}, SX].$$

Holonomies

Corollary

Let $h : \mathbb{S}^1 \to M_N(\mathbb{C})$ be a smooth path of self-adjoint operators such that:

(i) The spectrum $h(t)$ is symmetric.

(ii) For $\lambda_i(t), -\lambda_i(t) \in \sigma(h(t))$, the orthogonal projection

$$Q_i = P_i + P_{-i}$$

is constant in time, where $P_{\pm i}$ denote the respective spectral projections.

Then the spectrum of the holonomy U_h for the smooth path S_β in the Uhlmann's fibration coincides with the spectrum of the holonomy U_B for the smooth path P_{S_β} in the Berry's fibration.

Further Questions

(i) To what extent can the hypothesis

$$[\dot{X}_\beta(t), S_\beta(t)X_\beta(t)] = 0,$$

stated in our result be relaxed?

(ii) How might the perturbation of one differential equation by the other affect the spectrum of the holonomy?

(iii) Could the above restriction be associated with a topological constraint?

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