Calculus of variations for curves and surfaces - Problems

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1 Geodesics and energy

Let $\mathcal{N} \subset \mathbb{R}^N$ be an isometrically immersed Riemannian manifold. If $z \in \mathcal{N}$ and $T_z\mathcal{N}$ is the tangent space to $\mathcal{N}$ at z, then we define the symmetric $N \times N$-matrix field P_T over $\mathcal{N}$ such that

$$z \in \mathcal{N} \mapsto P_T(z) \in \mathrm{Sym}_{N\times N}(\mathbb{R}), \quad \text{the matrix of the orthogonal projection } \mathbb{R}^N \to T_z\mathcal{N}.$$

Problem 1.1. *Prove that if $\gamma : \mathbb{S}^1 \to \mathcal{N}$ is a closed curve such that for all $W : \mathbb{S}^1 \to \mathbb{R}^N$ such that $W(\theta) \in T_{\gamma(\theta)}\mathcal{N}$ for each θ there holds*

$$\int W \cdot \frac{\partial^2\gamma}{\partial\theta^2} d\theta = 0,$$

then γ satisfies the equation $\partial_\theta^2\gamma + \partial_\theta\left(P_T \circ \gamma\right)\partial_\theta\gamma = 0$, or more precisely, for each $i = 1, \ldots, N$ if γ^i is the i-th coordinate of γ, there holds

$$\partial_\theta^2\gamma^i(\theta) + \sum_{j=1}^N \partial_\theta\left(P_T^{ij}(\gamma(\theta))\right)\partial_\theta\gamma^j(\theta) = 0, \quad \forall\theta \in \mathbb{S}^1.$$

Problem 1.2. *Prove that γ is a geodesic in constant-speed parameterization (i.e. such that $|\partial_\theta\gamma|$ is constant along $\mathbb{S}^1$),* **if and only if** *γ is a critical point of the functional $\mathcal{E}(\gamma) := \int_{\mathbb{S}^1} |\partial_\theta\gamma|^2 d\gamma$ under perturbations within $\mathcal{N}$. More precisely, the latter means that*

$$\frac{d}{dt}\,\mathcal{E}\left(\pi_{\mathcal{N}}\left(\gamma + tV\right)\right)|_{t=0} = 0,$$

forall V a vector field in $C^\infty(\mathbb{S}^1, \mathbb{R}^N)$, and where $\pi_{\mathcal{N}}$ is the projection onto $\mathcal{N}$, which is well-defined over a small neighborhood of $\mathcal{N}$ in $\mathbb{R}^N$.

Hints:

- Use the result of the previous problem. Also note that $d\pi_{\mathcal{N}}(z) = P_T(z)$ for all $z \in \mathcal{N}$. Then use the chain rule together with the above expression of the critical point condition for $\mathcal{E}$.
- Justify and use the pointwise calculation

$$\left\langle\partial_\theta\gamma(\theta), P_T(\gamma(\theta))\partial_\theta^2\gamma(\theta)\right\rangle = \left\langle P_T(\gamma(\theta))\partial_\theta\gamma(\theta), \partial_\theta^2\gamma(\theta)\right\rangle = \left\langle\partial_\theta\gamma(\theta), \partial_\theta^2\gamma(\theta)\right\rangle,$$

in order to find that $\partial_\theta|\partial_\theta\gamma|^2 \equiv 0$.

2 Birkhoff curve-shortening process

Fix a large $Q \in \mathbb{N}$ and fix a compact submanifold $\mathcal{N} \subset \mathbb{R}^N$. We define the subspace

$$\Lambda^Q := \{\gamma : \mathbb{S}^1 \to \mathcal{N} : \ \exists x_1, \ldots, x_Q \in \mathbb{S}^1, \ \forall i, \ \gamma_{[x_i, x_{i+1}]} \text{ geodesic}\} \subset W^{1,2}(\mathbb{S}^1, \mathcal{N}),$$

where we define $x_{Q+1} := x_1$. Let also $G^Q \subset \Lambda^Q$ be the set of immersed closed geodesics in M with length at most $2\pi Q$.

Birkhoff's curve-shortening map $\Psi : \Lambda^Q \to \Lambda^Q$ is defined as follows:

- For $\gamma \in \Lambda^Q$, fix $x_0, \ldots, x_{2Q} = x_0 \in \mathbb{S}^1$.
- For $I_k := [x_{2k}, x_{2k+2}]$ replace $\gamma|_{I_k}$ by the shortest geodesic from $\gamma(x_{2k})$ to $\gamma(x_{2(k+1)})$. Repeat this for $k = 0, \ldots, Q-1$, obtaining a curve $\gamma^{(1)} \in \Lambda^Q$
- For $J_k := [x_{2k+1}, x_{2k+3}]$, replace $\gamma^{(1)}|_{J_k}$ by the shortest geodesic from $\gamma^{(1)}(x_{2k+1})$ to $\gamma^{(1)}(x_{2k+3})$. Repeating this for $k = 0, \ldots, Q-1$ gives a curve $\gamma^{(2)} \in \Lambda^Q$.
- Reparameterize $\gamma^{(2)}$ by arclength, fixing $\gamma^{(2)}(x_0)$, and define $\Psi(\gamma)$ to be the resulting curve.

Problem 2.1. *Prove the following properties of the map Ψ defined above:*

1. *(**Continuity**) Ψ is continuous with respect to the $W^{1,2}(\mathbb{S}^1, \mathcal{N})$ topology.*
2. *(**Curve shortening property**) $\Psi(\gamma)$ is homotopic to γ and $\mathcal{L}(\Psi(\gamma)) \le \mathcal{L}(\gamma)$, where $\mathcal{L} : \Lambda^Q \to \mathbb{R}^+$ is the length function.*
3. *(**$W^{1,2}$-distance bound**) There exists a continuous $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$ and*
$$\mathrm{dist}^2_{W^{1,2}}(\gamma, \Psi(\gamma)) \le \phi\left(\frac{\mathcal{L}(\gamma)^2}{\mathcal{L}(\Psi(\gamma))^2} - 1\right).$$
4. *(**Strict shortening for distant curves**) For each $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathrm{dist}_{W^{1,2}}(\gamma, G^Q) \ge \epsilon$*

Hints: We suggest to work at each step with constant speed curves (i.e. change coordinates by $s : \mathbb{S}^1 \to \mathbb{S}^1$ such that the curve $\tilde{\gamma}$ given by $\tilde{\gamma}(s(t)) = \gamma(t)$ has speed $|\tilde{\gamma}'| = \frac{\mathcal{L}(\gamma)}{2\pi}$).

1. It suffices to consider only the $W^{1,2}$-continuity of the first operation in the definition of Ψ, i.e. the substitution over all intervals I_k.
   - Prove Wirtinger's inequality: for $f \in W^{1,2}([0, Q], \mathcal{N})$ with $f(0) = f(Q)$, there holds $\int_0^Q |f|^2 \, dt \le (Q/2\pi)^2 \int_0^Q |f'|^2 \, dt$. (Prove this first for $Q = 1$ and first for $\mathcal{N} = \mathbb{R}^n$, and then apply modifications for the case of general $\mathcal{N}$.)
   - Use Wirtinger's inequality at each step, applied to suitable f, to prove that if $\gamma, \bar{\gamma}$ are close in $W^{1,2}$, then $\gamma^{(1)}, \bar{\gamma}^{(1)}$ are also close.
2. Construct an explicit homotopy by taking the geodesic arcs in $\mathcal{N}$ between $\gamma(x)$ and $\Psi(\gamma)(x)$. Use the triangle inequality.
3. A proof could go through the following steps

- Prove the following claim: For I an interval of length $\leq 2\pi/L$ and $\sigma_1 :\to \mathcal{N}$ Lipschitz curve with speed $|\sigma_1'| \leq L$, if $\sigma_2 : I \to \mathcal{N}$ is a minimizing geodesic with the same endpoints as σ_1, prove

$$\mathrm{dist}^2_{W^{1,2}}(\sigma_1, \sigma_2) \leq C(\mathcal{E}(\sigma_1) - \mathcal{E}(\sigma_2)), \quad \text{where} \quad \mathcal{E}(\gamma) := \int_{\mathbb{S}^1} |\gamma'|^2 \, d\theta.$$

For this, use again Wirtinger in order to reduce to estimating $\int |(\sigma_1 - \sigma_2)'|^2$. Then *justify the following computations*, valid for suitable C, C' depending on $\mathcal{N}$, and where $v^\perp$ is v's normal component to $\mathcal{N}$:

$$\begin{aligned}\int |\sigma_1'|^2 - \int |\sigma_2'|^2 - \int |(\sigma_1 - \sigma_2)'|^2 &= -2\int \langle \sigma_1 - \sigma_2, \sigma_2'' \rangle \\ &\leq C \int |(\sigma_1 - \sigma_2)^\perp||\sigma_2'| \leq C \int |\sigma_1 - \sigma_2|^2 \int |\sigma_2'|^2 \\ &\leq C' \int |\sigma_1 - \sigma_2|^2.\end{aligned}$$

The last term can be estimated by the L^2-norm of $(\sigma_1 - \sigma_2)'$ by Wirtinger again.

- Prove that $\mathcal{L}(\gamma)^2 \leq 2\mathcal{E}(\gamma)$, with equality if $|\gamma'|$ is constant. Then apply this to compare the changes of length before and after the substitutions by geodesic segments applied for constructing $\Psi(\gamma)$, to the $W^{1,2}$-distance of $\gamma, \Psi(\gamma)$. Do the estimates for each I_k separately and then repeat for the J_k.
- The final estimate should hold for $\phi(z) = z + \sqrt{z}$.

4. The last point can be proved by contradiction: if a sequence γ_j satisfies $\mathrm{dist}_{W^{1,2}}(\gamma_j, G) \geq \epsilon > 0$ but $\mathcal{E}(\Psi(\gamma_j)) \geq \mathcal{E}(\gamma_j) - 1/j$, prove that up to subsequence, $\gamma_j \to \gamma$ with $\gamma \in \Lambda$ and $\Psi(\gamma) = \gamma$, and reach a contradiction with this.

Problem 2.2. *Prove Fenchel's theorem:*

If $\gamma : \mathbb{S}^1 \to \mathbb{R}^m$ is a closed curve and $d\ell_\gamma$ is the arclength measure over γ and κ is its curvature, then

$$\int_{\mathbb{S}^1} |\kappa| d\ell_\gamma \geq 2\pi.$$

3 Stone's theorem, Hausdorff, paracompact and normal spaces

Problem 3.1. *Prove that every metric space is paracompact (Stone's theorem).*

Problem 3.2. *Prove that every Hausdorff paracompact space is normal (Prop. I.1 of the course).*

Problem 3.3. *Let $\mathcal{M}$ be a paracompact Banach manifold and ϕ a chart for $\mathcal{M}$ mapping an open set $\mathcal{O} \subset \mathcal{M}$ homeomorphically onto a Banach space V. Show by a counterexample that if B is a closed ball in V then $\phi^{-1}(B)$ needs not be closed in $\mathcal{M}$.*

Hint: If ℓ_2 is the Hilbert space of all real-valued sequences $x = (x_1, x_2, \ldots)$ such that $\|x\|_{\ell_2} := \sum |x_k|^2 < \infty$, then there exists a C^∞ diffeomorphism $\phi : \ell_2 \setminus \{0\} \to \ell_2$ which equals the identity outside the unit ball (result due to C. Bessaga, "*Every infinite dimensional Hilbert space is diffeomorphic with its unit sphere*", Bull. Acad. Polon. Sci., **14** (1966), 27-31).

4 Finsler manifolds

Problem 4.1. *Prove Theorem I.2 of the course. Namely, if $(\mathcal{M}, \|\cdot\|)$ is a Finsler manifold and we define a "minimum-length distance" (the* Palais distance*) between $p, q \in \mathcal{M}$ by*

$$d(p,q) := \inf\left\{\int_0^1 \|\omega'(t)\|_{\omega(t)}dt :\ \omega \in C^1([0,1],\mathcal{M}), \omega(0) = p, \omega(1) = q\right\},$$

then d defines a distance over $\mathcal{M}$ and as a metric space $(\mathcal{M}, d)$ has a topology equivalent with the Banach manifold topology of $\mathcal{M}$.

Hints:

1. Prove that if $\mathcal{M}$ is a Banach manifold, $\phi : U \to V$ is a chart for $\mathcal{M}$ and $\|\cdot\|$ is a norm for V, then let

$$\begin{aligned}
\overline{B(x_0,r)} &:= \{x \in U :\ \|\phi(x) - \phi(x_0)\| \leq r\},\\
B(x_0,r) &:= \{x \in U :\ \|\phi(x) - \phi(x_0)\| < r\},\\
S(x_0,r) &:= \{x \in U :\ \|\phi(x) - \phi(x_0)\| = r\}.
\end{aligned}$$

   then for $r > 0$ small enough $\overline{B(x_0,r)}$ is a closed neighborhood of x_0 in M, of which $B(x_0,r)$ is the interior and $S(x_0,r)$ is the boundary relative to M, and $S(x_0,r)$ separates $B(x_0,r)$ from $\mathcal{M} \setminus \overline{B(x_0,r)}$ in the sense that any continuous path between these two sets must cross $S(x_0,r)$.

2. In a chart as in the previous point, use the continuity of the Finsler norm and the fact that the Palais distance convergence can be tested by sequences of paths, to conclude.

5 Geometric Banach and Finsler manifolds, Palais-Smale condition

Let $\Sigma, \mathcal{N}$ are manifolds of dimensions k, m respectively, with $\mathcal{N}$ isometrically embedded into $\mathbb{R}^N$. In general, the Sobolev space $W^{1,p}(\Sigma,\mathcal{N})$ may be defined based on the fact that $W^{1,p}(\Sigma,\mathbb{R}^N)$ is a well-defined Banach space, and thus we can write

$$W^{1,p}(\Sigma,\mathcal{N}) := W^{1,p}(\Sigma,\mathbb{R}^N) \cap \{\phi :\ \phi(x) \in \mathcal{N} \text{ for a.e. } x \in \Sigma\}.$$

But in certain cases we may equivalently define $W^{1,p}(\Sigma,\mathcal{N})$ chart by chart, using the manifold structure of $\Sigma, \mathcal{N}$. In particular, we can do this whenever $W^{1,p}(\mathbb{R}^k,\mathbb{R}^N)$ embeds continuously into $C^0(\mathbb{R}^k,\mathbb{R}^N)$. This condition is ensured by the Sobolev embedding theorem if $p > k$.

Problem 5.1.
- *Show that $W^{1,2}(\mathbb{S}^1,\mathcal{N})$ has a Hilbert manifold structure.*
- *Let the Hilbert manifold $\mathcal{M} := W^{1,2}(\mathbb{S}^1,\mathcal{N})$ be endowed with the scalar product associated to the $W^{1,2}$-norm*

$$\|w\|_u := \|w\|_{W^{1,2}} \quad \text{if } w \text{ is a } W^{1,2}\text{-section of the pull-back bundle } u^{-1}(T\mathcal{N}).$$

  Show that this defines a Finsler manifold structure over X. Show that X is complete with respect to the Palais distance.

Problem 5.2.
- *Show that over $W^{1,2}(\mathbb{S}^1,\mathcal{N})$ the Dirichlet energy functional defined by $\mathcal{E}(\gamma) := \int_{\mathbb{S}^1} |\partial_\theta\gamma|^2 d\theta$ satisfies the Palais-Smale condition.*

- *Show that the Dirichlet energy functional* $\mathcal{E}(u) := \int_{\mathbb{S}^2} |Du|^2$ *over the Hilbert space* $W^{1,2}(\mathbb{S}^2, \mathbb{R}^3)$ *does not satisfy the Palais-Smale condition.*

Problem 5.3. *Prove Proposition I.5 of the course: For* $p > 1$, *over* $\mathcal{M} = W^{2,p}_{imm}(\mathbb{S}^1, \mathcal{N})$ *we have defined a Finsler structure by*

$$\|v\|_\gamma^p := \int_{S^1} \left(|\nabla^2 v|^2_{g_\gamma} + |\nabla v|^2_{g_\gamma} + |v|^2 \right) d\ell_{g_\gamma}.$$

Prove that $(\mathcal{M}, \|\cdot\|)$ *is complete with respect to the Palais distance.*

Why do we need the assumption $p > 1$*?*

Hint: Think how to adapt the proof of Prop. I.4 to this case.

Problem 5.4. *Prove that every* C^1*-function over a Finsler manifold admits a pseudo-gradient.*