EXERCISES

Geometrical aspects of spectral theory

16 October 2019

1. Show that every orthonormal sequence in a Hilbert space weakly converges to zero. (Recall that a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is weakly converging to zero if $(\phi, \psi_n) \to 0$ for every $\phi \in \mathcal{H}$).

2. Study low-lying **Dirichlet eigenvalues of the square**.
   (a) Determine the point spectrum of the Dirichlet Laplacian in the square of side π.
   (b) Identify the first eleven lowest eigenvalues (counting multiplicities) and arrange them in a non-decreasing order.
   (c) What is the highest multiplicity of these eigenvalues?

3. Study the case of **Neumann boundary conditions**, that is, consider the boundary-value problem

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in} \quad \Omega\,, \\ \dfrac{\partial\psi}{\partial n} = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1}$$

   where n denotes the outward unit normal vector field of $\partial\Omega$.
   (a) Find the eigenvalues and eigenfunctions if $\Omega := (0, a)$ is a segment of width $a > 0$.
   (b) Find the eigenvalues and eigenfunctions if $\Omega := (0, a) \times (0, b)$ is a rectangle of sides a and b.

4. Let $\Omega_1 \subset \Omega_2$ be bounded domains in $\mathbb{R}^d$ and let $\{\lambda_k^D(\Omega_1)\}_{k=1}^\infty$ and $\{\lambda_k^D(\Omega_2)\}_{k=1}^\infty$ denote the eigenvalues of the Dirichlet Laplacian arranged in a non-decreasing sequence counting multiplicities. Using the variational formula for the ground-state energy and the trivial extension, we have seen that $\lambda_1^D(\Omega_1) \geq \lambda_1^D(\Omega_2)$. More generally, one has $\lambda_k^D(\Omega_1) \geq \lambda_k^D(\Omega_2)$ for every $k \geq 1$; this property is called the **monotonicity of Dirichlet eigenvalues** (larger domain has smaller eigenvalues).

   Find a **counterexample to the monotonicity of Neumann eigenvalues**. More specifically, find domains $\Omega_1 \subset \Omega_2$ such that $\lambda_k^N(\Omega_1) < \lambda_k^N(\Omega_2)$ for some $k \geq 1$.
   Hint 1: Notice that the lowest eigenvalue $\lambda_1^N(\Omega)$ is always zero for any domain Ω, so you should look at $k = 2$ (or higher).
   Hint 2: Consider inscribed rectangles.

5. Let Ω be a local deformation of the straight strip $\Omega_0 := \mathbb{R} \times (0, 1)$ (that is, there exists a rectangle Q such that $\Omega \setminus Q = \Omega_0 \setminus Q$). Show that $[\pi^2, \infty) \subset \sigma(-\Delta_D^\Omega)$.
   Hint: Show that the sequence formed by $\psi_n(x, y) := \varphi((x - n)/n)\sin(\pi y)$, where $\varphi \in C_0^2((0, \infty))$ is a suitably normalised function, is an approximate eigenfunction.