Spring School in Analysis and Mathematical Physics

Santiago October 2019 Instructor : Serge Richard

Operator algebras: what are they good for ?

Exercise 1 (Unitization) *Let $\mathcal{A}$ be a Banach algebra, with norm $\|\cdot\|$, and consider $\tilde{\mathcal{A}} := \mathcal{A} \oplus \mathbb{C}$ as a vector space. On $\tilde{\mathcal{A}}$ we define the multiplication*

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu).$$

Check that $\tilde{\mathcal{A}}$ is a unital algebra with unit $(0, 1)$. Check that the map $\mathcal{A} \ni a \mapsto (a, 0) \in \tilde{\mathcal{A}}$ is an injective homomorphism, and that the image of this map is an ideal in $\tilde{\mathcal{A}}$. This allows us to identify $\mathcal{A}$ as an ideal in $\tilde{\mathcal{A}}$.

We usually write $a + \lambda$ for (a, λ). Check that the map $\tilde{\mathcal{A}} \ni a + \lambda \mapsto \lambda \in \mathbb{C}$ is a homomorphism with kernel $\mathcal{A}$. If we endow $\tilde{\mathcal{A}}$ with the map $\tilde{\mathcal{A}} \ni a + \lambda \mapsto \|a + \lambda\| := \|a\| + |\lambda| \in \mathbb{R}_+$, check that is a norm on $\tilde{\mathcal{A}}$.

Exercise 2 (Double centraliser) *Consider a C^*-algebra $\mathcal{A}$ and a pair (L, R) of bounded linear maps on $\mathcal{A}$ satisfying for any $a, b \in \mathcal{A}$*

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \qquad R(a)b = aL(b).$$

Such a pair is called a double centraliser*. For example, for any $c \in \mathcal{A}$, the pair (L_c, R_c) with $L_c(a) = ca$ and $R_c(a) = ac$ is a double centralizer.*

1) Check that $\|L\| = \|R\|$ (you can use that $\|a\| = \sup_{\|b\|\leq 1} \|ab\| = \sup_{\|b\|\leq 1} \|ba\|$).

We denote the set of double centralizers by $M(\mathcal{A})$ and endow it with the norm $\|(L, R)\| := \|L\| = \|R\|$. For (L_1, R_1) and (L_2, R_2) in $M(\mathcal{A})$ we define the product

$$(L_1, R_1)(L_2, R_2) := (L_1L_2, R_2R_1).$$

We also endow $M(\mathcal{A})$ with a map $* : (L, R) \mapsto (L, R)^* := (R^*, L^*)$ with $L^*(a) = \big(L(a^*)\big)^*$ and similarly for R.

2) Show that $M(\mathcal{A})$ is a algebra with this multiplication,

3) Show that $*$ defines an involution on $M(\mathcal{A})$,

4) Show that $M(\mathcal{A})$ with this multiplication, involution, and norm, is a unital C^*-algebra.

Observe that the map $\mathcal{A} \ni a \mapsto (L_a, R_a) \in M(\mathcal{A})$ is an isometric $*$-isomorphism, which allows us to identify $\mathcal{A}$ with its image in $M(\mathcal{A})$. With this identification, $\mathcal{A}$ becomes an ideal in $M(\mathcal{A})$.

Exercise 3 (The Grothendieck construction) *In this exercise, we show how one can associate an Abelian group to every Abelian semigroup (in a way analogous to how one obtains $\mathbb{Z}$ from $\mathbb{N}$).*

Let $(S, +)$ be an Abelian semigroup. Define a relation $\sim$ on $S \times S$ by $(x_1, y_1) \sim (x_2, y_2)$ if there exists $z \in S$ with $x_1 + y_2 + z = x_2 + y_1 + z$.

1) *Show that $\sim$ is an equivalence relation on $S \times S$.*

We write $G(S)$ for $(S \times S)/\sim$, and let $\langle x, y\rangle$ denote the equivalence class in $G(S)$ containing $(x, y) \in S \times S$. The operation

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$$

is well defined and turns $\big(G(S), +\big)$ into an Abelian group. Note that $-\langle x, y\rangle = \langle y, x\rangle$, and $\langle x, x\rangle = 0$ for all $x, y \in S$. The group $G(S)$ is called the Grothendieck group *of S.*

Fix $y \in S$ and consider the map $\gamma_S : S \ni x \mapsto \langle x + y, y\rangle \in G(S)$.

2) *Show that γ_S is independent of the choice of y,*

3) *Show that $G(S) = \{\gamma_S(x) - \gamma_S(y) \mid x, y \in S\}$,*

4) *Show that $\gamma_S(x) = \gamma_S(y)$ if and only if $x + z = y + z$ for some $z \in S$,*

5) *Show that γ_S is injective if and only if S has the* cancellation property*, namely whenever $x + z = y + z$ for $x, y, z \in S$ one has $x = y$.*

Exercise 4 (Whitehead lemma) *Let $\mathcal{A}$ be a unital C^*-algebra, and let u, v be unitary elements in $\mathcal{A}$. Let us also denote by $\sim_h$ the homotopy equivalence in the set $\mathcal{U}_2(\mathcal{A})$ (the set of unitary 2×2 matrices with entries in $\mathcal{A}$). Then*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

In particular one has

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the proof, one can first show that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and then use identities of the form:

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

References

Master class lecture notes available on

http://www.math.nagoya-u.ac.jp/~richard/lecture_notes.html

Further reading

- R. Douglas, *Banach algebra techniques in operator theory*, Second edition, Graduate Texts in Mathematics **179**, Springer-Verlag, New York, 1998.
- G. Murphy, *C^*-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990.
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- M. Roerdam, F. Larsen, N. Laustsen, *An introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts **49**, Cambridge University Press, Cambridge, 2000.